

# Coproduct Formula for Motivic Version of Yamamoto's Integral

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# Multiple zeta values (MZVs)

## Definition

**Multiple zeta value** is a real number defined by

$$\zeta(k_1, \dots, k_d) = \sum_{0 < n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \dots n_d^{k_d}},$$

where  $k_1, \dots, k_{d-1} \in \mathbb{Z}_{>0}$ ,  $k_d \in \mathbb{Z}_{>1}$ .

# iterated integral

## Definition

For  $a_0, \dots, a_{k+1} \in \mathbb{C}$  with  $a_0 \neq a_1, a_k \neq a_{k+1}$ , and a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  from  $a_0$  to  $a_{k+1}$  such that  $\gamma((0, 1)) \subset \mathbb{C} \setminus \{a_1, \dots, a_k\}$ , we define

$$I_\gamma(a_0; a_1, \dots, a_k; a_{k+1})$$

by the iterated integral

$$\int_{0 < t_1 < \dots < t_k < 1} \prod_{j=1}^k \frac{d\gamma(t_j)}{\gamma(t_j) - a_j} \in \mathbb{C}.$$

## Remark

MZVs are iterated integrals with  $a_i \in \{0, 1\}$ ,  $a_0 = 0, a_1 = 1, a_k = 0, a_{k+1} = 1$  and  $\gamma(t) = t$ . More precisely,

# Yamamoto's integral

## Definition (labeled poset)

$X = (X, \preceq_X, \delta_X)$  is called a **labeled poset** if  $X$  is a finite set,

$$\preceq_X \subset X \times X$$

is a partial order, and

$$\delta_X : X \rightarrow \mathbb{C}$$

is a map (this map is called a **labeling map** of  $X$ ). We denote the corresponding strict partial order of  $\preceq_X$  by  $\prec_X$ , and when the  $X$  is clear from the context, we denote  $\preceq_X$ ,  $\prec_X$  and  $\delta_X$  simply by  $\preceq$ ,  $\prec$  and  $\delta$ , respectively.

# Yamamoto's integral

## Definition (Yamamoto's integral)

For a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  such that  $\gamma((0, 1)) \subseteq \mathbb{C} \setminus \delta(X)$ , we say that  $X$  is **admissible** if  $\delta(x) \neq \gamma(1)$  for any maximal elements of  $X$ , and  $\delta(x) \neq \gamma(0)$  for any minimal elements of  $X$ . For an admissible labeled poset  $X$ , **Yamamoto's integral** is defined by

$$I_\gamma(X) := \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(\gamma(t_x)),$$

where

$$\Delta(X) := \{t = (t_x)_{x \in X} \in (0, 1)^X \mid t_x < t_y \text{ if } x \prec y\},$$

$$\omega_a(t) := \frac{dt}{t - a}.$$

# Property of Yamamoto's integral

## Definition

For  $(X, \preceq_X, \delta_X)$  and  $(Y, \preceq_Y, \delta_Y)$ , we define the **direct sum**

$$(X \sqcup Y, \preceq_{X \sqcup Y}, \delta_{X \sqcup Y}),$$

where

$$a \preceq_{X \sqcup Y} b \iff \begin{cases} a \preceq_X b \text{ and } a, b \in X \\ a \preceq_Y b \text{ and } a, b \in Y, \end{cases}$$

and

$$\delta_{X \sqcup Y}(a) = \begin{cases} \delta_X(a) \text{ if } a \in X \\ \delta_Y(a) \text{ if } a \in Y, \end{cases}$$

# Property of Yamamoto's integral

## Definition

A labeled poset  $(X, \preceq, \delta)$  is said to be **irreducible** if

$$(X, \preceq, \delta) \neq (Y, \preceq|_Y, \delta|_Y) \sqcup (X \setminus Y, \preceq|_{X \setminus Y}, \delta|_{X \setminus Y}),$$

for all  $\emptyset \subsetneq Y \subsetneq X$ , where  $\preceq|_Y$  is  $\preceq \cap Y^2$ .

## Definition

For  $(X, \preceq, \delta)$  and incomparable elements  $a, b \in X$ , we define

$$X_a^b := (X, \preceq \cup \{(x, y) \mid x \preceq a \text{ and } b \preceq y\}, \delta).$$

# Properties of Yamamoto's integral

## Definition

For  $(X, \preceq_X, \delta)$ , we define

$$\text{Tot}(X) = \{(X, \preceq, \delta) \mid \preceq_X \subset \preceq \text{ and } \preceq \text{ is a total order}\}.$$

## Proposition

Let  $X, Y$  be two labeled posets. For a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  such that  $\gamma((0, 1)) \subseteq \mathbb{C} \setminus (\delta_X(X) \cup \delta_Y(Y))$  and  $a, b \in X$  not comparable, we have

- 1  $I_\gamma(X)I_\gamma(Y) = I_\gamma(X \sqcup Y).$
- 2  $I_\gamma(X) = I_\gamma(X_a^b) + I_\gamma(X_b^a).$
- 3  $I_\gamma(X) = \sum_{Y \in \text{Tot}(X)} I_\gamma(Y).$

# Motivic iterated integrals

For  $k \geq 0$ ,  $a_0, \dots, a_{k+1} \in \mathbb{Q}$  and a piecewisely smooth path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  from  $a_0$  to  $a_{k+1}$  such that

$$\gamma((0, 1)) \subset \mathbb{C} \setminus \{a_1, \dots, a_k\} \text{ and } \gamma'(0), \gamma'(1) \in \mathbb{Q} \setminus \{0\},$$

one can define the motivic iterated integral

$$I_\gamma^m(a_0; a_1, \dots, a_k; a_{k+1}),$$

which is mapped to  $I_\gamma(a_0; a_1, \dots, a_k; a_{k+1})$  under the period map, and thus the motivic multiple zeta values are defined by

$$\zeta^m(k_1, \dots, k_d) := (-1)^d I^m(0; 1, \{0\}^{k_1-1}, \dots, 1, \{0\}^{k_d-1}; 1).$$

# Motivic version of Yamamoto's integral

## Definition

Let  $(X, \preceq, \delta)$  be a labeled poset with the labeling map

$$\delta : X \rightarrow \mathbb{Q}.$$

**Motivic version of Yamamoto's integral** is defined by

$$I_{\gamma}^{\text{m}}(X) := \sum_{Y \in \text{Tot}(X)} I_{\gamma}^{\text{m}}(Y),$$

where  $I_{\gamma}^{\text{m}}(Y)$  is the motivic iterated integral, which corresponds to iterated integral  $I_{\gamma}(Y)$ .

# Properties of motivic iterated integrals

## Proposition

- ①  $I_{\gamma}^{\mathfrak{m}}(a_0; a_1) = 1.$
- ②  $I_{\gamma}^{\mathfrak{m}}(a_0; a_1, \dots, a_k; a_{k+1}) = (-1)^k I_{\gamma^{-1}}^{\mathfrak{m}}(a_{k+1}; a_k, \dots, a_1; a_0).$
- ③ *Motivic iterated integrals  $I_{\gamma}^{\mathfrak{m}}(a; \alpha; b)$  satisfy shuffle relation i.e.*

$$I_{\gamma}^{\mathfrak{m}}(a; \alpha; b) \cdot I_{\gamma}^{\mathfrak{m}}(a; \beta; b) = I_{\gamma}^{\mathfrak{m}}(a; \alpha \sqcup \beta; b).$$

- ④ *Motivic iterated integrals  $I_{\gamma}^{\mathfrak{m}}(a; \omega; b)$  satisfy path composition formula i.e.*

$$I_{\gamma}^{\mathfrak{m}}(a; \omega; b) = \sum_{\alpha\beta=\omega} I_{\gamma'}^{\mathfrak{m}}(a; \alpha; c) \cdot I_{\gamma''}^{\mathfrak{m}}(c; \beta; b),$$

where  $\gamma$  is the composition of  $\gamma'$  and  $\gamma''$ .

# Coaction on $\mathcal{H}$

Let  $\mathcal{H} = \langle I_\gamma^m(a_0; a_1, \dots, a_k; a_{k+1}) \rangle_{\mathbb{Q}}$  be the  $\mathbb{Q}$ -algebra generated by the motivic version iterated integral, where  $I_\gamma^m(a_0; a_1, \dots, a_k; a_{k+1})$  runs over all cases considered in above. By shuffle relation  $\mathcal{H}$  becomes a subalgebra of the ring of periods of mixed Tate motives. Let  $\mathcal{A}$  be the quotient  $\mathcal{H}/(\mu)$  with natural projection  $\pi : \mathcal{H} \rightarrow \mathcal{A}$ , where  $\mu$  is the motivic version of  $2\pi i$  in  $\mathcal{H}$ . Goncharov proved a coproduct formula for motivic iterated integrals, which turns  $\mathcal{A}$  into a Hopf algebra, and Brown generalized the coproduct to the coaction on  $\mathcal{H}$ .

Coaction on  $\mathcal{H}$ 

## Theorem (Goncharov)

Let  $\Delta : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$  be the coaction on  $\mathcal{H}$ . Then,

$$\Delta(I_\gamma^m(a_0; a_1, \dots, a_k; a_{k+1})) = \sum_{s=0}^k \sum_{\substack{i_0 < \dots < i_{s+1} \\ i_0=0, i_{s+1}=k+1}} I_\alpha^m(a_{i_0}; a_{i_1}, \dots, a_{i_s}; a_{i_{s+1}}) \otimes \prod_{p=0}^s I^\alpha(a_{i_p}; a_{i_{p+1}}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

where

$$I^\alpha(a_{i_p}; a_{i_{p+1}}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}) := \pi(I_\gamma^m(a_{i_p}; a_{i_{p+1}}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}))$$

with an arbitrary path  $\gamma$  from  $a_{i_p}$  to  $a_{i_{p+1}}$ . Since  $I^\alpha$  does not depend on the choose of  $\gamma$ , this is well-defined.

# Infinitesimal coaction

## Definition

Let  $\pi'$  be the natural projection from  $\mathcal{A}$  to  $\mathcal{A}/\mathcal{A}_{>0}^2$ . For  $r \in \mathbb{N}$ , the infinitesimal coaction  $D_r : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$  is defined by

$$\begin{aligned} & D_r(I_\gamma^m(a_0; a_1, \dots, a_k; a_{k+1})) \\ &= \sum_{s=0}^{k-r} I^l(a_s; a_{s+1}, \dots, a_{s+r}; a_{s+r+1}) \\ & \quad \otimes I_\gamma^m(a_0; a_1, \dots, a_s, a_{s+r+1}, \dots, a_k; a_{k+1}) \end{aligned}$$

where

$$I^l(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}) := \pi'(I^a(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}})).$$

## Definition

For a labeled poset  $X$  a path  $\gamma$ , we define the **extension** of  $X$  with respect to  $\gamma$  as

$$\tilde{X}_\gamma := (\tilde{X}_\gamma, \preceq_{\tilde{X}_\gamma}, \tilde{\delta}_{\tilde{X}_\gamma}),$$

where

$$\tilde{X}_\gamma := X \cup \{x_0, x_1\}$$

$$\preceq_{\tilde{X}_\gamma} := \preceq_X \cup \{(x_0, x) \mid x \in \tilde{X}_\gamma\} \cup \{(x, x_1) \mid x \in \tilde{X}_\gamma\}$$

$$\tilde{\delta}_{\tilde{X}_\gamma}(x) := \begin{cases} \delta_X(x) & \text{if } x \in X \\ \gamma(0) & \text{if } x = x_0 \\ \gamma(1) & \text{if } x = x_1 \end{cases}.$$

When the path  $\gamma$  is clear, we denote  $\tilde{X}_\gamma$  simply by  $\tilde{X}$ .

## Definition

For posets  $Y \subset X$ , we define the subsets

$$Y^{\prec X} := \bigcup_{y \in Y} \{x \in X \mid x \preceq y\} \setminus Y$$

$$Y^{\succ X} := \bigcup_{y \in Y} \{x \in X \mid x \succeq y\} \setminus Y$$

and their subsets

$$Y^{\rightarrow X} := \{p \in Y^{\prec X} \mid \nexists l \in Y^{\prec X} \text{ s.t. } p \prec l\}$$

$$Y^{\leftarrow X} := \{q \in Y^{\succ X} \mid \nexists u \in Y^{\succ X} \text{ s.t. } u \prec q\}.$$

## Remark

We will also use the following notations.

- 1 We denote  $\{y\}^{\leftarrow X}$  (resp.  $\{y\}^{\rightarrow X}$ ,  $\{y\}^{\rightarrow X}$ ,  $\{y\}^{\leftarrow X}$ ) by  $y^{\leftarrow X}$  (resp.  $y^{\rightarrow X}$ ,  $y^{\rightarrow X}$ ,  $y^{\leftarrow X}$ ).
- 2 We denote  $x \in Y^{\rightarrow X}$  (resp.  $x \in Y^{\leftarrow X}$ ) by  $x \rightarrow_X Y$  or  $Y \leftarrow_X x$  (resp.  $x \leftarrow_X Y$  or  $Y \rightarrow_X x$ ).
- 3 When  $Y^{\rightarrow X}$  (resp.  $Y^{\leftarrow X}$ ) is a singleton we denote its unique element by  $\rightarrow_X Y$  (resp.  $\leftarrow_X Y$ ).

## Definition

Let  $X$  be a labeled poset. Fix  $Y \subset X$ . For  $p \rightarrow Y, q \leftarrow Y$ , we define

$$X_{(p, \hat{Y}, q)} := (X_{\hat{Y}}, \preceq_{X_{(p, \hat{Y}, q)}}, \delta_{X_{\hat{Y}}})$$

where

$$X_{\hat{Y}} := X \setminus Y$$

$$\begin{aligned} \preceq_{X_{(p, \hat{Y}, q)}} := & \preceq_X \setminus \{(a, b) \in \preceq_X \mid a \in Y \text{ or } b \in Y\} \\ & \cup \{(a, b) \in X_{\hat{Y}} \times X_{\hat{Y}} \mid a \in Y^{<X}, p \preceq_X b\} \\ & \cup \{(a, b) \in X_{\hat{Y}} \times X_{\hat{Y}} \mid b \in Y^{>X}, a \preceq_X q\} \end{aligned}$$

$$\delta_{X_{\hat{Y}}} := \delta_X|_{X_{\hat{Y}}}$$

## Remark

When  $|Y \rightarrow X| = |Y \leftarrow X| = 1$ , we denote  $X_{(p, \widehat{Y}, q)}$  simply by  $X_{\widehat{Y}}$ .  
 We denote  $X_{(p, \widehat{\{y\}}, q)}$  (resp.  $X_{\widehat{\{y\}}}$ ) simply by  $X_{(p, \widehat{y}, q)}$  (resp.  $X_{\widehat{y}}$ ).

## Definition

Let  $X$  be a labeled poset. For  $r \in \mathbb{N}$ , we define

$$X_r := \{Y \subset X \mid |Y| = r, \forall x \in (X \setminus Y) \nexists y, y' \in Y \text{ s.t. } y \preceq x \preceq y'\}.$$

## Definition

Let  $X$  be a labeled poset. For  $Y \in X_r$  and  $\overline{Y} \in \text{Tot}(Y)$ , we define

$$\text{Tot}_{\overline{Y}}(X) := \{\overline{X} \in \text{Tot}(X) \mid \overline{Y} \in \overline{X}_r, \preceq_{\overline{Y}} \subset \preceq_{\overline{X}}\}$$

# Definition of $D_r$

## Definition

When  $\preceq$  is a total order, using our notation to rewrite the definition of  $D_r$ . Then,

$$D_r : \mathcal{H} \rightarrow \mathcal{A}/\mathcal{A}_{>0}^2 \otimes \mathcal{H}.$$

by

$$D_r(I_\gamma^m(X)) := \sum_{Y \in X_r} I_{(\rightarrow_{\tilde{x}} Y; Y \rightarrow_{\tilde{x}})}^l(Y) \otimes I_\gamma^m(X_{\tilde{Y}})$$

where

$$I_{(p;q)}^l(Y) = \pi' \circ \pi(I_\gamma^m(Y)) \text{ with arbitrary path } \gamma \text{ from } \tilde{\delta}(p) \text{ to } \tilde{\delta}(q).$$

## Main theorem (1)

## Theorem (1)

Let  $X$  be a labeled poset and  $\gamma$  be a path. Then

$$D'_r(I_\gamma^m(X)) = \sum_{\substack{Y \in X_r \\ Y \text{ is irr.}}} \sum_{\substack{p \rightarrow_{\tilde{X}} Y \\ q \leftarrow_{\tilde{X}} Y}} I_{(p;q)}^l(Y) \otimes I_\gamma^m(X_{(p, \hat{Y}, q)}).$$

## Sketch of proof.

Step1. By the definition of motivic version of Yamamoto's integral,

$$D'_r(I_\gamma^m(X)) = \sum_{Y \in \text{Tot}(X)} D'_r(I_\gamma^m(Y)).$$

## Main theorem (1)

## Sketch of proof.

Step2. By path composition formula, modulo product, we have

$$I_{\gamma}^l(a; \omega; b) = \sum_{\alpha\beta=\omega} I_{\gamma'}^l(a; \alpha; c) \cdot I_{\gamma''}^l(c; \beta; b) = I_{\gamma'}^l(a; \omega; c) + I_{\gamma''}^l(c; \omega; b).$$

Using formula above to get

$$D'_r(I_{\gamma}^m(X)) = \sum_{Y \in X_r} \sum_{\substack{p \rightarrow Y \\ q \leftarrow Y}} \left( \prod_{i=1}^{n_Y} I_{(p; q)}^l(Y_i) \right) \otimes \left( \sum_{W \in \text{Tot}(X_{(p, \hat{Y}, q)})} I_{\gamma}^m(W) \right).$$

Step3. Again by the definition of motivic version of Yamamoto's integral and modulo product to finish the proof.

## Definition

Let  $X$  be a labeled poset. For  $x, x' \in X$ , we say  $x'$  is **adjacent** to  $x$  if

$$x \rightarrow x' \text{ or } x' \rightarrow x,$$

and denote it as  $x \leftrightarrow x'$ . For  $y \in Y \subset X$ , we define the **connected component** of  $y$  in  $Y$  as

$$C_{X,Y}(y) := \{y' \in Y \mid \exists y_1, \dots, y_n \in Y \text{ s.t. } y = y_1 \leftrightarrow \dots \leftrightarrow y_n = y'\}$$

and the **component set** of  $Y$  as

$$C_X(Y) := \{C_{X,Y}(y) \mid y \in Y\}.$$

Notice that  $C_X(Y)$  forms a partition of  $Y$ .

# Definition of $\Delta$

## Theorem

When  $\preceq_X$  is a total order, using our notation to rewrite the theorem of Brown. Then,

$$\Delta : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$$

can be rewritten as

$$\Delta(I_\gamma^m(X)) := \sum_{Y \subset X} \prod_{Z \in C_X(Y)} I_{(\rightarrow_{\bar{X}} Z; Z \rightarrow_{\bar{X}})}^a(Z) \otimes I_\gamma^m(X_{\hat{Y}})$$

# Definition of $\Delta$

## Definition

Let  $X$  be a labeled poset. For  $Y \subset X$ , we define

$$\Delta_Y(I_\gamma^m(X)) := \sum_{\bar{X} \in \text{Tot}(X)} \prod_{Z \in C_{\bar{X}}(Y)} I_{(\rightarrow_{\bar{X}} Z; Z \rightarrow_{\bar{X}})}^a(Z) \otimes I_\gamma^m(\bar{X}_{\hat{Y}})$$

## Remark

Let  $X$  be a labeled poset and  $\gamma$  be a path, we have the following relation.

$$\Delta(I_\gamma^m(X)) = \sum_{Y \subset X} \Delta_Y(I_\gamma^m(X))$$

## Main theorem (2)

## Theorem (2)

Let  $X$  be a labeled poset,  $\gamma$  be a path and  $Y \subset X$ . If for all  $Z \in C_X(Y)$  we have  $Z \in X_{|Z|}$  and  $|Z^{\rightarrow X}| = |Z^{\leftarrow X}| = 1$ , then one has

$$\Delta_Y(I_\gamma^m(X)) = \prod_{Z \in C_X(Y)} I_{(p_Z; q_Z)}^a(Z) \otimes I_\gamma^m(X_{\widehat{Y}}),$$

where  $p_Z := \rightarrow_{\widehat{X}} Z$  and  $q_Z := \leftarrow_{\widehat{X}} Z$ .

# Main theorem (2)

## Sketch of proof.

Step1. By the definition of motivic version of Yamamoto's integral,

$$\Delta(I_\gamma^m(X)) = \sum_{Y \in \text{Tot}(X)} \Delta(I_\gamma^m(Y)).$$

Step2. By condition  $Z \in X|_Z$ ,  $|Z \rightarrow X| = |Z \leftarrow X| = 1$  and path composition formula, we have

$$\Delta_Y(I_\gamma^m(X)) = \sum_{W \in \text{Tot}(X_{\hat{Y}})} \prod_{V \in C_X(Y)} I_{(p_V; q_V)}^a(V) \otimes I_\gamma^m(W).$$

Step3. Again by the definition of motivic version of Yamamoto's integral to finish the proof.

# Examples of Theorem(2)

Let us consider the case

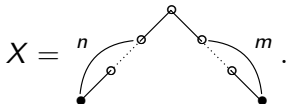
$$X = \{x_1, \dots, x_n, x_{n+1}, x'_1, \dots, x'_m\}$$

$$\preceq = \{(x_a, x_b) \mid a \leq b\} \cup \{(x'_a, x'_b) \mid a \leq b\} \cup \{(x'_a, x_{n+1}) \mid a \in \{1, \dots, m\}\}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x \in \{x_1, x'_1\} \\ 0 & \text{others} \end{cases}$$

$$\gamma(t) = t, \quad t \in [0, 1]$$

In other words,



## Examples of Theorem(2)

By Proposition of Yamamoto's integral, we have

$$I^m(X) = \sum_{i=1}^n \binom{n-i+m}{n-i} \zeta^m(i, n-i+m+1) \\ + \sum_{j=1}^m \binom{n+m-j}{m-j} \zeta^m(j, n+m-j+1).$$

## Examples of Theorem(2)

Let us calculate  $\Delta(I^m(X))$  using Theorem (2). First, using the proposition of motivic iterated integral, it is easy to show the formula

$$I^m(0; \{0\}^a, 1, \{0\}^b; 1) = (-1)^{a+1} \binom{a+b}{a} \zeta^m(a+b+1),$$

and using  $\binom{i}{k} = \binom{i-1}{k} + \binom{i-1}{k-1}$ , it is easy to show the formula

$$\sum_{k=0}^i (-1)^{i-k+1} \frac{m}{m+k} \binom{i}{k} = \frac{(-1)^{i+1}}{\binom{i+m}{m}}.$$

Next, using the remark of coproduct to have

$$\Delta(I^m(X)) = \sum_{Y \subset X} \Delta_Y(I^m(X)).$$

## Examples of Theorem(2)

Motivic iterated integrals have the following property

$$I^m(0; 0; 1) = I^m(0; 1; 1) = I^m(a; a_1, \dots, a_k; a) = 0. \quad (k \geq 1)$$

Finally using Theorem (2), Proposition of Yamamoto's integral, and property of Motivic iterated integrals to determine two types of  $Y$ , which may have non-zero terms of  $\Delta_Y(I^m(X))$ .

$$\textcircled{1} \quad Y_{1,i} = \{x_i, \dots, x_{n+1}, x'_1, \dots, x'_m\}.$$

$$\textcircled{2} \quad Y_{2,j} = \{x_1, \dots, x_{n+1}, x'_j, \dots, x'_m\}.$$

Since in this case, the integral is MZVs, hence we can write the formula of  $\Delta(I^m(X))$  by two types of  $Y$  and in terms of MZVs

$$\begin{aligned} \Delta(I^m(X)) &= 1 \otimes I^m(X) + I^a(X) \otimes 1 \\ &+ \sum_{i=3}^{n+1} I^a(Y_{1,i}) \otimes (-\zeta^m(i-1)) + \sum_{j=3}^{m+1} I^a(Y_{2,j}) \otimes (-\zeta^m(j-1)). \end{aligned}$$

## Examples of Theorem(2)

By Proposition of Yamamoto's integral and the formula above we have

$$\begin{aligned}
 I^{\alpha}(Y_{1,n+1-i}) &= \sum_{k=0}^i \binom{m-1+k}{k} I^{\alpha}(0; \{0\}^{i-k}, 1, \{0\}^{m+k}; 1) \\
 &= \sum_{k=0}^i \frac{m}{m+k} \binom{m+k}{k} (-1)^{i-k+1} \binom{m+i}{i-k} \zeta^{\alpha}(m+i+1) \\
 &= \binom{m+i}{m} \left( \sum_{k=0}^i (-1)^{i-k+1} \frac{m}{m+k} \binom{i}{k} \right) \zeta^{\alpha}(m+i+1) \\
 &= \binom{m+i}{m} \frac{(-1)^{i+1}}{\binom{i+m}{m}} \zeta^{\alpha}(m+i+1) \\
 &= (-1)^{i+1} \zeta^{\alpha}(m+i+1)
 \end{aligned}$$

## Examples of Theorem(2)

Hence we get

$$\begin{aligned}\Delta(I^m(X)) &= 1 \otimes I^m(X) + I^a(X) \otimes 1 \\ &+ \sum_{i=0}^{n-2} (-1)^i \zeta^a(m+i+1) \otimes \zeta^m(n-i) \\ &+ \sum_{j=0}^{m-2} (-1)^j \zeta^a(n+j+1) \otimes \zeta^m(m-j).\end{aligned}$$

# Examples of Theorem(1)

Let us consider the case

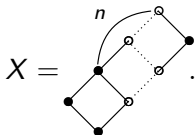
$$X = \{x_1, \dots, x_n, x'_1, \dots, x'_n\}$$

$$\preceq = \{(x_a, x_b) \mid a \leq b\} \cup \{(x'_a, x'_b) \mid a \leq b\} \cup \{(x_a, x'_b) \mid a \leq b\}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x \in \{x'_1, x'_2, x_1, x_n\} \\ 0 & \text{others} \end{cases},$$

$$\gamma(t) = t, \quad t \in [0, 1].$$

In other words,



# Examples of Theorem(1)

Let us calculate  $D_3(I^m(X))$  using Theorem (1). For convenience where we assume  $n \geq 5$  and let  $\delta(x_0) = 0, \delta(x'_{n+1}) = 1$ . First notice that the irreducible elements in  $X_3$  only have four types

- 1  $Y_{1,m} = \{x_m, x_{m+1}, x_{m+2}\}.$
- 2  $Y_{2,m} = \{x'_m, x_m, x_{m+1}\}.$
- 3  $Y_{3,m} = \{x'_m, x'_{m+1}, x_{m+1}\}.$
- 4  $Y_{4,m} = \{x'_m, x'_{m+1}, x'_{m+2}\}.$

# Examples of Theorem(1)

Thus, by Theorem (1) and the proposition of motivic iterated integrals we have

$$\begin{aligned}
 & D_3(I(X)) \\
 &= I_{(x_0;x'_1)}^l(Y_{1,1}) \otimes I^m(X_{(x_0,\widehat{Y}_{1,1},x'_1)}) + I_{(x_0;x'_2)}^l(Y_{2,1}) \otimes I^m(X_{(x_0,\widehat{Y}_{2,1},x'_2)}) \\
 &+ I_{(x'_1;x'_3)}^l(Y_{2,2}) \otimes I^m(X_{(x'_1,\widehat{Y}_{2,2},x'_3)}) + I_{(x'_1;x_4)}^l(Y_{2,2}) \otimes I^m(X_{(x'_1,\widehat{Y}_{2,2},x_4)}) \\
 &+ I_{(x_1;x_3)}^l(Y_{3,1}) \otimes I^m(X_{(x_1,\widehat{Y}_{3,1},x_3)}) + I_{(x'_1;x_4)}^l(Y_{3,2}) \otimes I^m(X_{(x'_1,\widehat{Y}_{3,2},x_4)}) \\
 &+ I_{(x_{n-1};x'_{n+1})}^l(Y_{3,n-1}) \otimes I^m(X_{(x_{n-1},\widehat{Y}_{3,n-1},x'_{n+1})}) \\
 &+ I_{(x'_{n-2};x'_{n+1})}^l(Y_{3,n-1}) \otimes I^m(X_{(x'_{n-2},\widehat{Y}_{3,n-1},x'_{n+1})}).
 \end{aligned}$$

# Examples of Theorem(1)

$$I_{(x_0; x'_1)}^l(Y_{1,1}) \otimes I^m(X_{(x_0, \widehat{Y}_{1,1}, x'_1)}) =$$

$$I_{(x_0; x'_1)}^l \left( \begin{array}{c} \circ \\ / \backslash \\ \bullet \end{array} \right) \otimes I^m \left( \begin{array}{c} \circ \\ / \backslash \\ \bullet \end{array} \right) \cdot$$

## Theorem (Hirose, Murahara, and Onozuka)

If  $k$  is an admissible index such that  $k(i, j) = k(i + 1, j + 1)$  for all  $(i, j), (i + 1, j + 1) \in \lambda$ , then the SMZV  $\zeta(k)$  has Yamamoto's integral expression, i.e. we have

$$\zeta \left( \begin{array}{c} \boxed{k_i} \dots \boxed{k_1} \\ \dots \\ \boxed{k_r} \end{array} \right) = I \left( \begin{array}{c} k_1 \dots k_i \dots k_r \\ \text{diagrammatic representation} \end{array} \right),$$

where  $I_i$  is the number of  $k_i$ 's.

By Theorem above, we have

$$\zeta^m \left( \begin{array}{|c|c|} \hline n & 1 \\ \hline 1 & n \\ \hline \end{array} \right) = I^m \left( \begin{array}{c} \text{Diagram} \end{array} \right).$$

Therefore, our formula gives a cocation formula for the Schur MZV on the left-hand side.



## Proposition

$$\zeta^*(k_d, \dots, k_1) = I \left( \begin{array}{c} \text{Diagram with } d \text{ nodes and } k_r \text{ edges} \end{array} \right)$$

$$= \sum_{\substack{\mathbb{k}_1 \dots \mathbb{k}_r = \mathbb{k} \\ 1 \leq r \leq d}} (-1)^{d-r} \zeta(\mathbb{k}_1) \cdots \zeta(\mathbb{k}_r),$$

where  $\mathbb{k} = (k_1, \dots, k_d)$ .

Example ( $d=4, \zeta^*(5, 4, 3, 2)$ )

$$\begin{aligned} \zeta^*(5, 4, 3, 2) &= \zeta(2)\zeta(3)\zeta(4)\zeta(5) - \zeta(2, 3)\zeta(4)\zeta(5) \\ &\quad - \zeta(2)\zeta(3, 4)\zeta(5) - \zeta(2)\zeta(3)\zeta(4, 5) + \zeta(2, 3, 4)\zeta(5) \\ &\quad + \zeta(2, 3)\zeta(4, 5) + \zeta(2)\zeta(3, 4, 5) - \zeta(2, 3, 4, 5) \end{aligned}$$

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Thanks for your attention!