A Map Between Arborifications of Multiple Zeta Values

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Multiple zeta values

Multiple zeta values(MZVs)

Definition

Multiple zeta values are a real numbers defined by

$$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \prod_{i=1}^{d} \frac{1}{n_i^{k_i}},$$

where $k_1, \ldots, k_{d-1} \in \mathbb{Z}_{>0}, k_d \in \mathbb{Z}_{>1}$.

Iterated integrals

Definition

The **iterated integral** for $a_0, a_1, \ldots, a_k, a_{k+1} \in \mathbb{R}$

$$I(a_0; a_1, \ldots, a_k; a_{k+1}) := \int_{a_0 < t_1 < \cdots < t_k < a_{k+1}} \prod_{j=1}^{k} \omega_{a_j}(t_j),$$

is defined using differential 1-forms

$$\omega_a(t) := \frac{dt}{t-a}.$$

Remark

MZVs are iterated integrals with $a_i \in \{0, 1\}$,
$a_0 = 0, a_1 = 1, a_k = 0, a_{k+1} = 1$ and $\gamma(t) = t$. More precisely,

$$\zeta(k_1, \ldots, k_d) = (-1)^d I(0; 1, \{0\}^{k_1 - 1}, \ldots, 1, \{0\}^{k_d - 1}; 1).$$

Hopf algebra structure

Definition

- $\mathcal{X} := \{x_0, x_1\}$
  $\mathbb{Q}\langle\mathcal{X}\rangle :=$ non-commutative polynomial algebra generated by $\mathcal{X}$
  The triple $(\mathbb{Q}\langle\mathcal{X}\rangle, ш, \Delta)$ is a graded commutative Hopf algebra, where ш is the shuffle product and Δ is the coproduct.
- $\mathcal{Y} := \{y_n \mid n \in \mathbb{N}\}$
  $\mathbb{Q}\langle\mathcal{Y}\rangle :=$ non-commutative polynomial algebra generated by $\mathcal{Y}$
  The triple $(\mathbb{Q}\langle\mathcal{Y}\rangle, *, \Delta)$ is a graded commutative Hopf algebra, where $*$ is the stuffle product and Δ is the coproduct.

Remark

Here is an important correspondence.

$$\begin{array}{ccc} y_{n_1}\cdots y_{n_r} & \xrightarrow{\mathfrak{s}} & x_1x_0^{n_1-1}\cdots x_1x_0^{n_r-1} \\ \downarrow & & \downarrow \\ \sum_{0<m_1<\cdots<m_r}\prod_{i=1}^{r}\frac{1}{m_i^{n_i}} & & I(0;1,\{0\}^{n_1-1},\ldots,1,\{0\}^{n_r-1};1) \end{array}$$

The arborifications

Rooted trees and planar rooted trees

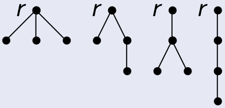
Definition

- A **rooted tree** $T = (T, r)$ is a tree $T = (V(T), E(T))$ in which one vertex r is designated as the root of the tree T, where $V(T)$ denotes the vertex set of T and $E(T)$ denotes the edge set of T.
- The **depth function** ρ_T is a function from $V(T)$ to $\mathbb{Z}_{\geq 0}$ which sends a vertex v to the length of the path from r to v.
- A **planar rooted tree** $T = (T, r, \alpha_T)$ is defined as a rooted tree (T, r) and a **total order relation** $\alpha_T \subset V(T) \times V(T)$ on the vertex set $V(T)$ which satisfies
  1. $\forall u, v \in V(T)\ \rho_T(u) < \rho_T(v) \Rightarrow (u, v) \in \alpha_T$,
  2. If $\{u, v\}, \{x, y\}$ in $E(T)$, $\rho_T(u) = \rho_T(x) = \rho_T(v) - 1 = \rho_T(y) - 1$ and $(u, x) \in \alpha_T$, then $(v, y) \in \alpha_T$.

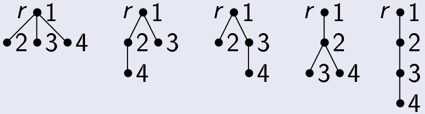
Rooted trees and planar rooted trees

Example

An example of (non-planar) rooted tree with $|V(T)| = 4$



An example of planar rooted tree with $|V(T)| = 4$



Rooted forests and planar rooted forests

Definition

A **forest** F of rooted trees (resp. planar rooted trees) is constructed by
step 1. Removing the root r from a rooted tree (resp. planar rooted tree) T.
step 2. Designating each vertex v that satisfies $\rho_T(v) = 1$ as the root of its connected component in the graph $T \setminus \{r\}$.
Note that a forest of planar rooted trees is equipped with a total order relation.

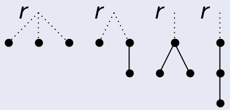
Remark

A forest F is a disjoint union of rooted trees.
In the planar case, the roots are ordered: $(r_i, r_j) \in \alpha$ whenever $i \leq j$.

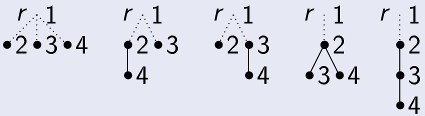
Rooted forests and planar rooted forests

Example

An example of (non-planar) rooted forest with $|V(T)| = 3$



An example of planar rooted forest with $|V(T)| = 3$



$\mathcal{D}$-decoration

Definition

Let $\mathcal{D}$ be a set. A $\mathcal{D}$-**decorated rooted tree (resp. forest)** is a rooted tree D (resp. forest F_D) together with a decoration map δ_D (resp. δ_{F_D}) from $V(D)$ (resp. $V(F_D)$) to $\mathcal{D}$.

Definition

The **opposite tree order** of a rooted tree T is a partial order relation $\preceq_T \subset V(T) \times V(T)$ on $V(T)$ which is defined as $(u, v) \in \preceq_T$ if and only if the path from r to u contains the path from r to v.

Remark

In this talk, the set $\mathcal{D}$ is $\mathcal{X}$ or $\mathcal{Y}$, where $\mathcal{X} := \{x_0, x_1\}$, $\mathcal{Y} := \{y_n \,|\, n \in \mathbb{N}\}$. We generalize the two definitions above to the planar case by the same way.

Arborified multiple zeta values of the first kind

Definition

Arborified multiple zeta values of the first kind are multiple zeta values associated with a $\mathcal{Y}$-decorated rooted tree Y (resp. forest), defined as the harmonic series associated to the triple $(V(Y), \preceq_Y, \delta_Y)$.

$$\zeta(Y) := \sum_{\substack{n_v \in \mathbb{N} \\ n_u < n_v \text{ if } u \prec_Y v}} \prod_{v \in V(Y)} \frac{1}{n_v^{k_v}},$$

where k_v is the integer n such that $\delta_Y(v) = y_n$.

Arborified multiple zeta values of the second kind

Definition

Arborified multiple zeta values of the second kind are multiple zeta values associated with a $\mathcal{X}$-decorated rooted tree X (resp. forest), defined as Yamamoto's integral associated to the triple $(V(X), \preceq_X, \delta_X)$.

$$\zeta(X) := I(X) = \int_{\Delta(X)} \prod_{v \in V(X)} \omega_{\delta_X(v)}(t_v),$$

where $\Delta(X) := \{\mathrm{t} = (t_v)_{v \in V(X)} \in (0,1)^{V(X)} \mid t_u < t_v \text{ if } u \prec_X v\}$, $\omega_{x_0}(t) := \frac{dt}{t}, \omega_{x_1}(t) := \frac{dt}{1-t}$.

Hopf algebra structure

Foissy (2002) proved the following two polynomial algebras have the Hopf algebra structure.

Definition

Let $\mathcal{D}$ be a set.

- Let $\mathbb{Q}[\mathcal{T}^{\mathcal{D}}]$ be the commutative polynomial algebra, where $\mathcal{T}^{\mathcal{D}}$ is the set of non-empty $\mathcal{D}$-decorated rooted trees.
  Butcher-Connes-Kreimer Hopf algebra (BCK Hopf algebra) of $\mathcal{D}$-decorated rooted tree $\mathcal{H}_{BCK}^{\mathcal{D}}$ is defined as the triple $(\mathbb{Q}[\mathcal{T}^{\mathcal{D}}], \pi, \Delta)$ which is a graded non-commutative Hopf algebra with the product π and the coproduct Δ.
- Let $\mathbb{Q}\langle\mathcal{T}^{P\mathcal{D}}\rangle$ be the non-commutative polynomial algebra, where $\mathcal{T}^{P\mathcal{D}}$ is the set of non-empty $\mathcal{D}$-decorated planar rooted trees.
  non-commutative Butcher-Connes-Kreimer Hopf algebra (NBCK Hopf algebra) of $\mathcal{D}$-decorated planar rooted tree $\mathcal{H}_{NBCK}^{P\mathcal{D}}$ is defined as the triple $(\mathbb{Q}\langle\mathcal{T}^{P\mathcal{D}}\rangle, \pi, \Delta)$ which is a graded non-commutative Hopf algebra with the product π and the coproduct Δ.

Manchon's question (2020)

Manchon's question

Based on Foissy's work, Manchon introduced the simple arborification $\mathfrak{a}_{\mathcal{X}}$ and the contracting arborification $\mathfrak{a}_{\mathcal{Y}}$, which are Hopf algebra morphisms.

Question

Manchon posed the question to find a natural map $\mathfrak{s}^T$ with respect to the tree structures, which makes the following diagram commutative.

$$\begin{array}{ccc} \mathcal{H}_{BCK}^{\mathcal{Y}} & \xrightarrow{\mathfrak{s}^T} & \mathcal{H}_{BCK}^{\mathcal{X}} \\ \downarrow{\scriptstyle \mathfrak{a}_{\mathcal{Y}}} & & \downarrow{\scriptstyle \mathfrak{a}_{\mathcal{X}}} \\ \mathbb{Q}\langle\mathcal{Y}\rangle & \xrightarrow{\mathfrak{s}} & \mathbb{Q}\langle\mathcal{X}\rangle \end{array}$$

Manchon's question

Definition

The **ladder tree section** $\ell_{\mathcal{X}}$ of the simple arborification $\mathfrak{a}_{\mathcal{X}}$ (resp. $\ell_{\mathcal{Y}}$ of the contracting arborification $\mathfrak{a}_{\mathcal{Y}}$) is defined by

$$\ell_{\mathcal{X}}(x_{m_1}x_{m_2}\cdots x_{m_s}) = \begin{matrix} \bullet\ x_{m_s} \\ \vdots \\ \bullet\ x_{m_2} \\ | \\ \bullet\ x_{m_1} \end{matrix} \quad \left(\text{resp. } \ell_{\mathcal{Y}}(y_{n_1}y_{n_2}\cdots y_{n_t}) = \begin{matrix} \bullet\ y_{n_t} \\ \vdots \\ \bullet\ y_{n_2} \\ | \\ \bullet\ y_{n_1} \end{matrix}\right).$$

Note that ladder tree has a unique total order relation α, therefore $\ell_{\mathcal{X}}$ (resp. $\ell_{\mathcal{Y}}$) is also a section of $\mathfrak{a}_{P\mathcal{X}}$ (resp. $\mathfrak{a}_{P\mathcal{Y}}$). In this case, we use the notation $\ell_{P\mathcal{X}}$ and $\ell_{P\mathcal{Y}}$.

Manchon gave an obvious answer, which is given by

$$\mathfrak{s}^T = \ell_{\mathcal{X}} \circ \mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}.$$

It makes the diagram commutative, but has the drawback of completely destroying the geometry of trees.

Clavier's attempt

Definition

Let $\mathcal{D}$ be a set, and d be an element in $\mathcal{D}$. The **grafting operator** $B_+^d : \mathcal{H}_{BCK}^{\mathcal{D}} \to \mathcal{H}_{BCK}^{\mathcal{D}}$ (resp. $B_+^d : \mathcal{H}_{NBCK}^{P\mathcal{D}} \to \mathcal{H}_{NBCK}^{P\mathcal{D}}$) is an algebra morphism that maps any $\mathcal{D}$-decorated (resp. planar) rooted forest to a $\mathcal{D}$-decorated (resp. planar) rooted tree by grafting all components onto the common root decorated by d.

Definition

Let $Y = B_+^{y_n}(Y_1 \cdots Y_m)$ be a $\mathcal{Y}$-decorated rooted tree in $\mathcal{H}_{BCK}^{\mathcal{Y}}$. The linear map $\mathfrak{s}^N : \mathcal{H}_{BCK}^{\mathcal{Y}} \to \mathcal{H}_{BCK}^{\mathcal{X}}$ is defined recursively by

$$\mathfrak{s}^N(B_+^{y_n}(Y_1 \cdots Y_m)) = (B_+^{x_0})^n \circ B_+^{x_1}(\mathfrak{s}^N(Y_1 \cdots Y_m)),$$

where

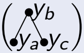
$$\mathfrak{s}^N(Y_1 \cdots Y_m) = \mathfrak{s}^N(Y_1) \cdots \mathfrak{s}^N(Y_m),$$

a forest of $\mathcal{X}$-decorated rooted trees.

Clavier's attempt

Example

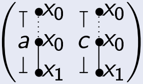
An example of the natural map $\mathfrak{s}^N$ of a $\mathcal{Y}$-decorated rooted tree.

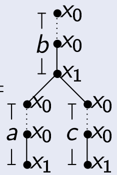
$\mathfrak{s}^N$  $= \mathfrak{s}^N\left(B_+^{y_b}(\bullet_{y_a}\bullet_{y_c})\right)$

$$= (B_+^{x_0})^{b-1} \circ B_+^{x_1}\left(\mathfrak{s}^N(\bullet_{y_a}\bullet_{y_c})\right)$$

$$= (B_+^{x_0})^{b-1} \circ B_+^{x_1}\left(\mathfrak{s}^N(\bullet_{y_a})\,\mathfrak{s}^N(\bullet_{y_c})\right)$$

$$= (B_+^{x_0})^{b-1} \circ B_+^{x_1}\left((B_+^{x_0})^{a-1} \circ B_+^{x_1}(\emptyset)\,(B_+^{x_0})^{c-1} \circ B_+^{x_1}(\emptyset)\right)$$

$= (B_+^{x_0})^{b-1} \circ B_+^{x_1}$ 

$=$ 

Clavier's attempt

Theorem (Clavier, 2020)

Let F be a forest in $\mathcal{H}_{BCK}^{\mathcal{Y}}$. If $\zeta(F)$ converges, then we have

$$\zeta(\mathfrak{s}^N(F)) \leq \zeta(F).$$

Furthermore, the equality holds if, and only if, F is a ladder forest.

The following diagram is non-commutative.

$$\begin{array}{ccc} \mathcal{H}_{BCK}^{\mathcal{Y}} & \xrightarrow{\mathfrak{s}^N} & \mathcal{H}_{BCK}^{\mathcal{X}} \\ \Big\downarrow{\scriptstyle \mathfrak{a}_{\mathcal{Y}}} & & \Big\downarrow{\scriptstyle \mathfrak{a}_{\mathcal{X}}} \\ \mathbb{Q}\langle \mathcal{Y} \rangle & \xrightarrow{\mathfrak{s}} & \mathbb{Q}\langle \mathcal{X} \rangle \end{array}$$

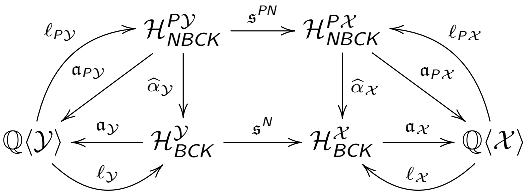
Lift of Manchon's question

Based on Foissy's work, we generalize our setting to the case of planar rooted trees.

Definition

The natural projection from NBCK Hopf algebra $\mathcal{H}^{P\mathcal{D}}_{NBCK} = \mathbb{Q}\langle \mathcal{T}^{P\mathcal{D}} \rangle$ to BCK Hopf algebra $\mathcal{H}^{\mathcal{D}}_{BCK} = \mathbb{Q}[\mathcal{T}^{\mathcal{D}}]$ by removing the total order relation is denoted by $\widehat{\alpha}_{\mathcal{D}}$, which is an algebra morphism.

The lifting map $\mathfrak{a}_{P\mathcal{X}}, \ell_{P\mathcal{X}}, \mathfrak{s}^{PN}, \mathfrak{a}^{P\mathcal{Y}}, \ell_{P\mathcal{Y}}$ are defined by such that the following diagram commutative.



Main result

The error term

From Clavier's theorem, we know that

$$\mathfrak{a}_{\mathcal{X}} \circ \mathfrak{s}^N(Y) \neq \mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}(Y).$$

To make the diagram commutative, it is sufficient to consider the error term

$$\ell_{\mathcal{X}}(\mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}(Y) - \mathfrak{a}_{\mathcal{X}} \circ \mathfrak{s}^N(Y)).$$

The map $\mathfrak{s}^T$ must send Y to $\mathfrak{s}^N(Y) + \ell_{\mathcal{X}}(\mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}(Y) - \mathfrak{a}_{\mathcal{X}} \circ \mathfrak{s}^N(Y))$. Let Y be the simplest non-ladder forest $B_+^{y_b}(B_+^{y_a}(\emptyset)\ B_+^{y_c}(\emptyset))$.
In this case, the error term $\ell_{\mathcal{X}}(\mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}(Y) - \mathfrak{a}_{\mathcal{X}} \circ \mathfrak{s}^N(Y))$ is given by

$$\mathfrak{s}^N\left(\begin{smallmatrix} \bullet\, y_b \\ | \\ \bullet\, y_{a+c} \end{smallmatrix} - \sum_{i=1}^{a-1} \binom{a-i+c-1}{c-1} \begin{smallmatrix} \bullet\, y_b \\ | \\ \bullet\, y_{a+c-i} \\ | \\ \bullet\, y_i \end{smallmatrix} - \sum_{i=1}^{c-1} \binom{c-i+a-1}{a-1} \begin{smallmatrix} \bullet\, y_b \\ | \\ \bullet\, y_{a+c-i} \\ | \\ \bullet\, y_i \end{smallmatrix} \right).$$

Note that this error term is determined by a rooted tree and two vertices cannot be compared in opposite tree order.

The error term

Definition

Let T be a planar rooted tree. The **minimal incomparable pair** of T is defined as minimal element of $V(T) \times V(T) \setminus \preceq_T$ with respect to the lexicographic order associated to α_T.

Definition

Let Y be a $\mathcal{Y}$-decorated planar rooted tree, and let (a, b) be the minimal incomparable pair of Y. The **error term** Y^e of a $\mathcal{Y}$-decorated planar rooted tree Y is defined as

$$
\begin{aligned}
& B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}} (B_+^{y_{n_a+n_b}} (F_a\, F_b)\, F) \\
& - \sum_{i=1}^{n_a-1} \binom{n_a - i + n_b - 1}{n_b - 1} B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}} (B_+^{y_{n_a+n_b-i}} (B_+^{y_i} (F_a)\, F_b)\, F) \\
& - \sum_{i=1}^{n_b-1} \binom{n_b - i + n_a - 1}{n_a - 1} B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}} (B_+^{y_{n_a+n_b-i}} (B_+^{y_i} (F_b)\, F_a)\, F).
\end{aligned}
$$

The process tree

Definition

The **process tree** pr(Y) **of the contracting arborification** $\mathfrak{a}_{P\mathcal{Y}}$ of a $\mathcal{Y}$-decorated planar rooted tree Y is a $\mathcal{H}^{P\mathcal{Y}}_{NBCK}$-decorated planar rooted tree defined recursively by

$$\mathrm{pr}(Y) = B_+^Y(\mathrm{pr}(Y_{a+b})\ \mathrm{pr}(Y_b^a)\ \mathrm{pr}(Y_a^b)),$$

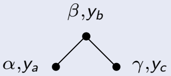
where (a, b) is the minimal incomparable pair of Y and

$$\begin{aligned} Y_{a+b} &:= B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}}(B_+^{y_{n_a+n_b}}(F_a\, F_b)\, F) \\ Y_b^a &:= B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}}(B_+^{y_{n_a}}(B_+^{y_{n_b}}(F_b)\, F_a)\, F) \\ Y_a^b &:= B_+^{y_{n_1}} \circ \cdots \circ B_+^{y_{n_m}}(B_+^{y_{n_b}}(B_+^{y_{n_a}}(F_a)\, F_b)\, F). \end{aligned}$$

The process tree

Example

Consider the $\mathcal{Y}$-decorated planar rooted tree

$Y =$ 

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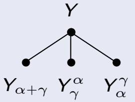
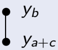
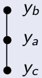
The process tree $\mathrm{pr}(Y)$ is given by

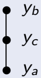
$\mathrm{pr}(Y) =$ 

,

where

$Y_{\alpha+\gamma} =$ 

, $Y^{\alpha}_{\gamma} =$ 

, $Y^{\gamma}_{\alpha} =$ 

.

Main theorem

Definition

The linear map $\phi : \mathcal{H}_{NBCK}^{P\mathcal{Y}} \to \mathcal{H}_{NBCK}^{P\mathcal{Y}}$ is defined by

$$\phi(Y) = Y + \sum_{v \in V(pr(Y))} \left(\delta_{\mathrm{pr}(Y)}(v)\right)^{e}.$$

The map $\mathfrak{s}^{PT}$ is defined by

$$\mathfrak{s}^{PT}(Y) = \mathfrak{s}^{PN} \circ \phi(Y).$$

Main theorem

Theorem (F.)

The following diagram is commutative.

$$\begin{array}{ccc} & \mathcal{H}^{P\mathcal{Y}}_{NBCK} & \\ \nearrow^{\phi} & & \searrow^{\mathfrak{s}^{PN}} \\ \mathcal{H}^{P\mathcal{Y}}_{NBCK} & \xrightarrow{\mathfrak{s}^{PT}} & \mathcal{H}^{P\mathcal{X}}_{NBCK} \\ \downarrow{\scriptstyle \mathfrak{a}_{P\mathcal{Y}}} & & \downarrow{\scriptstyle \mathfrak{a}_{P\mathcal{X}}} \\ \mathbb{Q}\langle\mathcal{Y}\rangle & \xrightarrow{\mathfrak{s}} & \mathbb{Q}\langle\mathcal{X}\rangle \end{array}$$

Answer to Manchon's question

Definition

Let Y be a $\mathcal{Y}$-decorated rooted tree and A_Y the set of all total order relations that can make Y a planar rooted tree. The order of this set $|A_Y|$ is given by

$$deg(r) \times \prod_{v \in V(Y) \setminus \mathrm{leaf}(Y) \setminus \{r\}} (deg(v) - 1).$$

The section $\beta_{\mathcal{Y}}$ of $\hat{\alpha}_{\mathcal{Y}}$ is defined by

$$\beta_{\mathcal{Y}}(Y) := \frac{1}{|A_Y|} \sum_{\alpha \in |A_Y|} Y_\alpha,$$

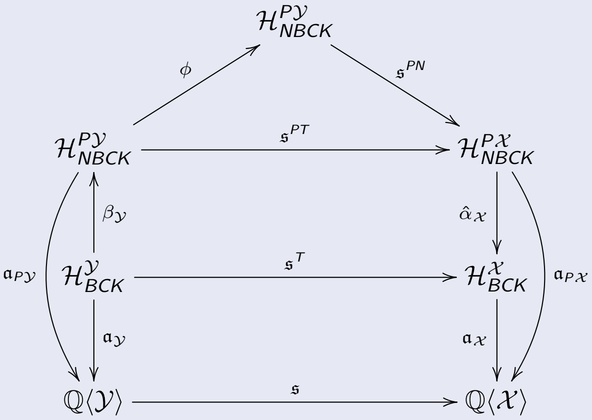
where Y_α is the $\mathcal{Y}$-decorated planar rooted tree obtained by equipping the $\mathcal{Y}$-decorated rooted tree Y with the total order relation $\alpha \in A_Y$. The map $\mathfrak{s}^T$ is defined by

$$\hat{\alpha}_{\mathcal{X}} \circ \mathfrak{s}^{PT} \circ \beta_{\mathcal{Y}}.$$

Answer to Manchon's question

Corollary (F.)

The following diagram is commutative.



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Thank you for your attention!