A Map Between Arborifications of Multiple Zeta Values

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2025 RIMS Symposia (open) “Various Aspects of Multiple Zeta Values”

2025/09/30

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Arborified multiple zeta values

The arborifications

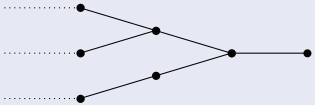
Example

1. total order

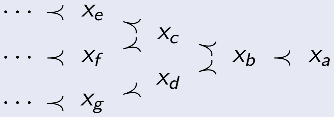
$$x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_{n-2} \prec x_{n-1} \prec x_n$$

2. ladder tree

3. rooted tree



4. partial order



Multiple zeta values(MZVs)

Definition (Multiple zeta values)

- $k_1, \dots, k_d \in \mathbb{N}, k_d > 1$

$$\zeta(k_1, \dots, k_d) := \sum_{0<n_1<\cdots<n_d} \prod_{i=1}^{d} \frac{1}{n_i^{k_i}}$$

Definition (Iterated integral)

- $a_0, a_1, \dots, a_k, a_{k+1} \in \{0, 1\}$
- $\omega_0(t) := \frac{dt}{t}, \omega_1(t) := \frac{dt}{t-1}$

$$I(a_0; a_1, \dots, a_k; a_{k+1}) := \int_{a_0<t_1<\cdots<t_k<a_{k+1}} \prod_{j=1}^{k} \omega_{a_j}(t_j)$$

Remark (iterated integral expression of MZVs)

$$\zeta(k_1, \dots, k_d) = (-1)^d I(0; 1, \{0\}^{k_1-1}, \dots, 1, \{0\}^{k_d-1}; 1)$$

Arborified multiple zeta values of the first kind

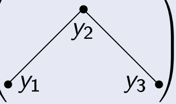
Definition

Arborified multiple zeta values of the first kind are multiple zeta values associated with a $\mathcal{Y}$-decorated rooted tree Y (resp. forest), defined as the harmonic series associated to the triple $(V(Y), \preceq_Y, \delta_Y)$.

$$\zeta(Y) := \sum_{\substack{n_v \in \mathbb{N} \\ n_u < n_v \text{ if } u \prec_Y v}} \prod_{v \in V(Y)} \frac{1}{n_v^{k_v}},$$

where k_v is the integer n such that $\delta_Y(v) = y_n$.

Example

ζ (
) $= \zeta(1,3,2) + \zeta(3,1,2) + \zeta(4,2)$

Arborified multiple zeta values of the second kind

Definition

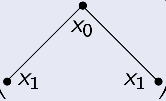
Arborified multiple zeta values of the second kind are multiple zeta values associated with a $\mathcal{X}$-decorated rooted tree X (resp. forest), defined as Yamamoto's integral associated to the triple $(V(X), \preceq_X, \delta_X)$.

$$\zeta(X) := I(X) = \int_{\Delta(X)} \prod_{v \in V(X)} \omega_{\delta_X(v)}(t_v),$$

where $\Delta(X) := \{\mathrm{t} = (t_v)_{v\in V(X)} \in (0,1)^{V(X)} \mid t_u < t_v \text{ if } u \prec_X v\}$, $\omega_{x_0}(t) := \frac{dt}{t}, \omega_{x_1}(t) := \frac{dt}{1-t}$.

Example

$$\zeta\left(\quad \right)$$



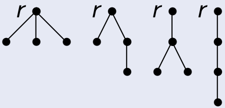
$$= I(0;1,1,0;1) + I(0;1,1,0;1) = 2\zeta(1,2)$$

Hopf algebra structure

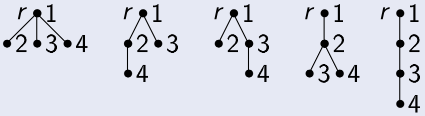
Rooted trees

Example (Rooted trees)

An example of (non-planar) **rooted tree** with $|V(T)| = 4$



An example of **planar rooted tree** with $|V(T)| = 4$



Hopf algebra structure

Definition (Hopf algebra for MZVs)

- $\mathcal{X} := \{x_0, x_1\}$
- $\mathbb{Q}\langle\mathcal{X}\rangle :=$ non-commutative polynomial algebra generated by $\mathcal{X}$

  $(\mathbb{Q}\langle\mathcal{X}\rangle, ⧢, \Delta)$: Hopf algebra with shuffle product for integral

- $\mathcal{Y} := \{y_n \mid n \in \mathbb{N}\}$
- $\mathbb{Q}\langle\mathcal{Y}\rangle :=$ non-commutative polynomial algebra generated by $\mathcal{Y}$

  $(\mathbb{Q}\langle\mathcal{Y}\rangle, *, \Delta)$: Hopf algebra with stuffle product for series

Remark (correspondence between integral and series)

$$\begin{array}{ccc} y_{n_1}\cdots y_{n_r} & \xrightarrow{\mathfrak{s}} & x_1 x_0^{n_1-1}\cdots x_1 x_0^{n_r-1} \\ \downarrow & & \downarrow \\ \sum_{0<m_1<\cdots<m_r} \prod_{i=1}^{r} \frac{1}{m_i^{n_i}} & & I(0;1,\{0\}^{n_1-1},\ldots,1,\{0\}^{n_r-1};1) \end{array}$$

Hopf algebra structure

Definition (Butcher-Connes-Kreimer Hopf algebra)

- $\mathcal{D}$: set
- $\mathcal{T}^{\mathcal{D}}$ (resp. $\mathcal{T}^{P\mathcal{D}}$) : set of non-empty $\mathcal{D}$-decorated (resp. planar) rooted trees
- $\mathbb{Q}[\mathcal{T}^{\mathcal{D}}]$ (resp. $\mathbb{Q}\langle\mathcal{T}^{P\mathcal{D}}\rangle$) : commutative (resp. non-commutative) polynomial algebra generated by $\mathcal{T}^{\mathcal{D}}$ (resp. $\mathcal{T}^{P\mathcal{D}}$)
- $\mathcal{H}_{BCK}^{\mathcal{D}} := (\mathbb{Q}[\mathcal{T}^{\mathcal{D}}], \pi, \Delta)$ (resp. $\mathcal{H}_{NBCK}^{P\mathcal{D}} := (\mathbb{Q}\langle\mathcal{T}^{P\mathcal{D}}\rangle, \pi, \Delta)$) : (resp. non-commutative) BCK Hopf algebra of $\mathcal{D}$-decorated (resp. planar) rooted tree

Example

$$\mathcal{H}_{BCK}^{\mathcal{X}} := (\mathbb{Q}[\mathcal{T}^{\mathcal{X}}], ⧢, \Delta) \text{ (resp. } \mathcal{H}_{NBCK}^{P\mathcal{X}} := (\mathbb{Q}\langle\mathcal{T}^{P\mathcal{X}}\rangle, ⧢, \Delta))$$

$$\mathcal{H}_{BCK}^{\mathcal{Y}} := (\mathbb{Q}[\mathcal{T}^{\mathcal{Y}}], *, \Delta) \text{ (resp. } \mathcal{H}_{NBCK}^{P\mathcal{Y}} := (\mathbb{Q}\langle\mathcal{T}^{P\mathcal{Y}}\rangle, *, \Delta))$$

Manchon's question (2020)

Manchon's question

Based on Foissy's work, Manchon introduced the simple arborification $\mathfrak{a}_{\mathcal{X}}$ and the contracting arborification $\mathfrak{a}_{\mathcal{Y}}$, which are Hopf algebra morphisms.

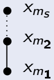
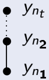
Question

Manchon posed the question to find a natural map $\mathfrak{s}^T$ with respect to the tree structures, which makes the following diagram commutative.

$$\begin{array}{ccc} \mathcal{H}_{BCK}^{\mathcal{Y}} & \xrightarrow{\ \mathfrak{s}^T\ } & \mathcal{H}_{BCK}^{\mathcal{X}} \\ {\scriptstyle \mathfrak{a}_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{a}_{\mathcal{X}}} \\ \mathbb{Q}\langle\mathcal{Y}\rangle & \xrightarrow{\ \mathfrak{s}\ } & \mathbb{Q}\langle\mathcal{X}\rangle \end{array}$$

Manchon's answer

Definition (ladder tree section)

$\ell_{\mathcal{X}}(x_{m_1} x_{m_2} \cdots x_{m_s}) :=$  $\left(\text{resp. } \ell_{\mathcal{Y}}(y_{n_1} y_{n_2} \cdots y_{n_t}) :=\right.$  $\left.\right)$.

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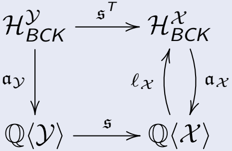
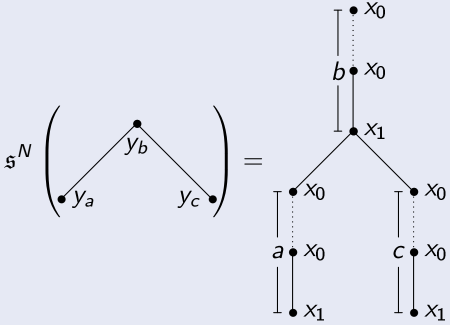
Answer (Manchon)

- $\mathfrak{s}^T := \ell_{\mathcal{X}} \circ \mathfrak{s} \circ \mathfrak{a}_{\mathcal{Y}}$: Non-natural map makes diagram commute.



Clavier's attempt

Example (natural map)



Clavier's attempt

Theorem (Clavier, 2020)

Let F be a forest in $\mathcal{H}_{BCK}^{\mathcal{Y}}$. If $\zeta(F)$ converges, then we have

$$\zeta(\mathfrak{s}^N(F)) \leq \zeta(F).$$

Furthermore, the equality holds if, and only if, F is a ladder forest.

Answer (Clavier)

- $\mathfrak{s}^T := \mathfrak{s}^N$: Natural map makes diagram non-commute.

$$\begin{array}{ccc} \mathcal{H}_{BCK}^{\mathcal{Y}} & \xrightarrow{\mathfrak{s}^T} & \mathcal{H}_{BCK}^{\mathcal{X}} \\ {\scriptstyle \mathfrak{a}_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{a}_{\mathcal{X}}} \\ \mathbb{Q}\langle\mathcal{Y}\rangle & \xrightarrow{\mathfrak{s}} & \mathbb{Q}\langle\mathcal{X}\rangle \end{array}$$

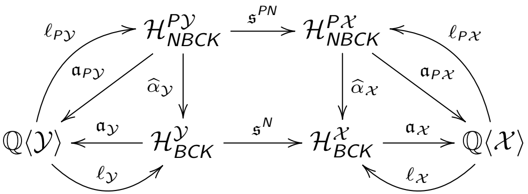
Lift of Manchon's question

Based on Foissy's work, we generalize our setting to the case of planar rooted trees.

Definition

The natural projection from NBCK Hopf algebra $\mathcal{H}_{NBCK}^{P\mathcal{D}} = \mathbb{Q}\langle \mathcal{T}^{P\mathcal{D}} \rangle$ to BCK Hopf algebra $\mathcal{H}_{BCK}^{\mathcal{D}} = \mathbb{Q}[\mathcal{T}^{\mathcal{D}}]$ by removing the total order relation is denoted by $\widehat{\alpha}_{\mathcal{D}}$, which is an algebra morphism.

The lifting map $\mathfrak{a}_{P\mathcal{X}}, \ell_{P\mathcal{X}}, \mathfrak{s}^{PN}, \mathfrak{a}^{P\mathcal{Y}}, \ell_{P\mathcal{Y}}$ are defined by such that the following diagram commutative.



Main result

Main theorem

Definition

- Y : $\mathcal{Y}$-decorated planar rooted tree
- δ_Y : decoration map, a map from $V(Y)$ to $\mathcal{Y}$
- Y^e : error term of Y, an element in $\mathcal{H}_{NBCK}^{P\mathcal{Y}}$
- $pr(Y)$: process tree of Y, a $\mathcal{H}_{NBCK}^{P\mathcal{Y}}$-decorated planar rooted tree

The linear map $\phi : \mathcal{H}_{NBCK}^{P\mathcal{Y}} \to \mathcal{H}_{NBCK}^{P\mathcal{Y}}$ is defined by

$$\phi(Y) = Y + \sum_{v \in V(pr(Y))} \left(\delta_{\mathrm{pr}(Y)}(v)\right)^e .$$

The map $\mathfrak{s}^{PT}$ is defined by

$$\mathfrak{s}^{PT}(Y) = \mathfrak{s}^{PN} \circ \phi(Y).$$

Main theorem

Theorem (F.)

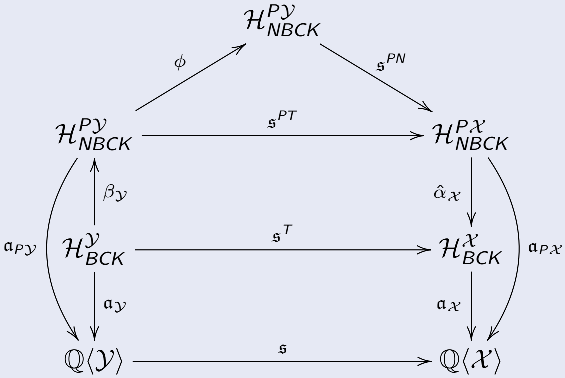
The following diagram is commutative.

$$
\begin{array}{ccccc}
 & & \mathcal{H}^{P\mathcal{Y}}_{NBCK} & & \\
 & \overset{\phi}{\nearrow} & & \overset{\mathfrak{s}^{PN}}{\searrow} & \\
\mathcal{H}^{P\mathcal{Y}}_{NBCK} & & \xrightarrow{\quad \mathfrak{s}^{PT} \quad} & & \mathcal{H}^{P\mathcal{X}}_{NBCK} \\
\Big\downarrow {\scriptstyle \mathfrak{a}_{P\mathcal{Y}}} & & & & \Big\downarrow {\scriptstyle \mathfrak{a}_{P\mathcal{X}}} \\
\mathbb{Q}\langle\mathcal{Y}\rangle & & \xrightarrow{\quad \mathfrak{s} \quad} & & \mathbb{Q}\langle\mathcal{X}\rangle
\end{array}
$$

Answer to Manchon's question

Corollary (F.)

Let $\beta_{\mathcal{Y}}$ be the section of $\hat{\alpha}_{\mathcal{Y}}$. The following diagram is commutative.



References

References

- Fan, Ku-Yu, *A map between arborifications of multiple zeta values*, arXiv:2508.20387 [math.NT], 2025, doi:10.48550/arXiv.2508.20387, https://arxiv.org/abs/2508.20387

Thank you for your attention!