p-adic multiple L-functions and twisted multiple Bernoulli numbers

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Overview

Furusho and Jarossay's work

Theorem

For any r-tuple $(n_i)_r$ of positive integers, the family

$$\left(p^{\sum_{i=1}^{r} n_i} L_{p,r}((n_i)_r;(\omega^{-n_i})_r;(1)_r;c)\right)_{p\in\mathcal{P}_c}$$

is expressed as

$$\sum_{\mathrm{l}=(l_i)_r\in\mathbb{N}_0^r}\ \sum_{\substack{\epsilon\in\mu_c\\ (\epsilon_i)_r\in(\mu_c\setminus\{1\})^r}}\ \sum_{J=(P_1,P_2,P_3)\in E_r}\ \sum_{\substack{\xi\in\mu_c\\ \prod_{j\in P_3}\left(\frac{\epsilon_j^{(P_{2,3})}}{\epsilon_j}\right)^{-\kappa_j'}\xi=\epsilon}}$$

$$\frac{\epsilon}{\prod_{i=1}^{r}(1-\epsilon_i)}\prod_{i=1}^{r}\binom{-n_i}{l_i}\mathcal{B}_{0,\xi}^{(\mathrm{l}_J,\boldsymbol{\epsilon}_J,\boldsymbol{\kappa}_J)}\sum_{\delta\in\Delta(T_{r,J})}\mathfrak{H}(\mathrm{w}(\mathrm{l})_\delta)^{\mathrm{Frob}^{-1}}.$$

$$p\text{-adic } L\text{-functions} = \sum(\text{a variant of CMBNs})\times(\text{CMHVs})$$

Furusho, Komori, Matsumoto, and Tsumura's work

Theorem

Let $\xi_1, \ldots, \xi_r \in \mathbb{C}$ be roots of unity and $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \le j \le r)$. Assume that

$$\xi_j \neq 1 \text{ for all } j \ (1 \le j \le r).$$

Then, with the above notation, the generalized Euler–Zagier–Lerch type complex multiple zeta functions $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be analytically continued to $\mathbb{C}^r$ as an entire function in (s_j). For $n_1, \ldots, n_r \in \mathbb{N}_0$,

$$\zeta_r((-n_j);(\xi_j);(\gamma_j)) = (-1)^{r+n_1+\cdots+n_r}\mathfrak{B}((n_j);(\xi_j^{-1});(\gamma_j)).$$

zeta functions of $(-n_j)$ = TMBNs of (n_j)

My work

Lemma

The following formula holds for every l, ξ.

$$\mathcal{B}_{l,\xi}^{((l_i)_r;(\epsilon_i)_r;(\kappa_i)_{r-1})} = \sum_{\mathbf{e}=(e_i)_r\in\{0\}\times\{0,1\}^{r-1}} \sum_{\substack{\forall 1\le i\le j\le r\ m_{\mathbf{e},i,j}\in\mathbb{N}_0 \\ m_{\mathbf{e},1,1}\le l_1 \\ \cdots \\ m_{\mathbf{e},1,r}+\cdots+m_{\mathbf{e},r,r}\le l_r}} \sum_{i_1=0}^{1-e_1}\cdots\sum_{i_r=0}^{1-e_r}$$

$$(-1)^{\sum_{k=1}^r (e_k+i_k)} \prod_{k=1}^r (\kappa_{k-1}+i_k)^{\sum_{j=k}^r m_{\mathbf{e},k,j}}$$

$$\prod_{k=1}^r \binom{l_k}{m_{\mathbf{e},1,k},\ldots,m_{\mathbf{e},k,k},\ l_k-\left(\sum_{i=1}^k m_{\mathbf{e},i,k}\right)} \delta_{\xi,\prod_{k=1}^r \epsilon_k^{-(\kappa_{k-1}+i_k)}} \delta_{l,\sum_{k=1}^r \sum_{j=k}^r m_{\mathbf{e},k,j}}$$

$$\mathfrak{B}_{(l_1-m_{\mathbf{e},1,1}\circ_{e_2}\cdots\circ_{e_r} l_r-(\sum_{i=1}^r m_{\mathbf{e},i,r}))}(\epsilon_1^{-1}(1-e_1)\circ_{e_2}\cdots\circ_{e_r}\epsilon_r^{-1}(1-e_r))$$

(a variant of CMBNs) $= \sum$(TMBNs)

Main theorem

$$
\begin{aligned}
&p\text{-adic } L\text{-functions}\\
&=\sum(\text{a variant of CMBNs}) \times (\text{CMHVs}) \text{ [FJ]}\\
&=\sum(\text{TMBNs}) \times (\text{CMHVs}) \text{ (F.)}\\
&=\sum(\text{complex zeta functions}) \times (\text{CMHVs}) \text{ [FKMT]}
\end{aligned}
$$

Corollary

For any $n \in \mathbb{N}$, the family

$$\left(p^n L_{p,1}(n;\omega^{-n};1;c)\right)_{p\in\mathcal{P}_c}$$

is expressed as

$$\sum_{l=0}^{\infty} \sum_{\epsilon\in\mu_c\setminus\{1\}} \binom{-n}{l} (-1)^{1+l} \zeta_1\left(-l;\epsilon^{-1};1\right) \mathfrak{H}(n+l,\epsilon^{-1})^{\mathrm{Frob}^{-1}}.$$

Generalization of Bernoulli numbers

Twisted Bernoulli numbers and Bernoulli numbers

Definition

For any root of unity ξ, we define the *twisted Bernoulli numbers* by

$$\mathcal{H}(t;\xi) = \frac{1}{1-\xi e^t} = \sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi)\frac{t^n}{n!},$$

where we formally let $(-1)! = 1$.

Example

In the case $\xi = 1$, we have

$$\mathfrak{B}_{-1}(1) = -1,\ \mathfrak{B}_n(1) = -\frac{B_{n+1}}{n+1}\ (n \in \mathbb{N}_0).$$

In the case $\xi \neq 1$, we have

$$\mathfrak{B}_0(\xi) = \frac{1}{1-\xi},\ \mathfrak{B}_1(\xi) = \frac{\xi}{(1-\xi)^2},\ \mathfrak{B}_2(\xi) = \frac{\xi(\xi+1)}{(1-\xi)^3},\dots$$

Twisted multiple Bernoulli numbers (TMBNs)

Definition

Let $r \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ and let $\xi_1, \ldots, \xi_r \in \mathbb{C} \setminus \{1\}$ be roots of unity. Set

$$\mathcal{H}_r((t_j);(\xi_j);(\gamma_j)) := \prod_{j=1}^{r} \mathcal{H}\left(\gamma_j \left(\sum_{k=j}^{r} t_k\right); \xi_j\right) = \prod_{j=1}^{r} \frac{1}{1-\xi_j \exp(\gamma_j \sum_{k=j}^{r} t_k)}$$

and define *twisted multiple Bernoulli numbers* $\mathfrak{B}((n_j);(\xi_j);(\gamma_j))$ for $(n_j) \in \mathbb{Z}_{\geq 0}^r$ by

$$\mathcal{H}_r((t_j);(\xi_j);(\gamma_j)) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \mathfrak{B}((n_j);(\xi_j);(\gamma_j)) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}.$$

Generalized Euler-Zagier-Lerch type multiple zeta-function

Definition

Let $\xi_1, \ldots, \xi_r \in \mathbb{C}$ be roots of unity. For $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leq j \leq r)$, where $\Re$ means the real part of the complex number. The *multiple zeta-function of the generalized Euler-Zagier-Lerch type* is defined by

$$\zeta_r((s_j); (\xi_j); (\gamma_j)) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{j=1}^{r} \frac{\xi_j^{m_j}}{(m_1\gamma_1 + \cdots + m_j\gamma_j)^{s_j}},$$

which is absolutely convergent in the region

$$\mathcal{D}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r \,|\, \Re(s_{r-k+1} + \cdots + s_r) > k \ (1 \leq k \leq r)\}.$$

The case $\gamma_1 = \cdots = \gamma_r = 1$

Theorem

Let $\xi_1, \ldots, \xi_r \in \mathbb{C}$ be roots of unity and $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leq j \leq r)$. Assume that

$$\xi_j \neq 1 \text{ for all } j \ (1 \leq j \leq r).$$

Then, with the above notation, $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be analytically continued to $\mathbb{C}^r$ as an entire function in (s_j). For $n_1, \ldots, n_r \in \mathbb{N}_0$,

$$\zeta_r((-n_j);(\xi_j);(\gamma_j)) = (-1)^{r+n_1+\cdots+n_r}\mathfrak{B}((n_j);(\xi_j^{-1});(\gamma_j)).$$

Remark

In the case $r = 1$, we have $\mathfrak{B}(n;\xi;1) = \mathfrak{B}_n(\xi)$. In the general case, we use the notation $\mathfrak{B}_{(n_j)_r}((\xi_j)_r) := \mathfrak{B}((n_j)_r;(\xi_j)_r;(1)_r)$ when $\gamma_1 = \cdots = \gamma_r = 1$. Hence, we have

$$\zeta_r((-n_j);(\xi_j);(1)) = (-1)^{r+n_1+\cdots+n_r}\mathfrak{B}_{(n_j)_r}((\xi_j^{-1})_r).$$

Cyclotomic multiple harmonic sums (CMHSs)

Definition

Let

$$c \in \mathbb{Z}_p^\times \cap \mathbb{N}_{\geq 2}, \qquad (l_i)_r = (l_1, \dots, l_r) \in \mathbb{N}_0^r,$$
$$(\epsilon_i)_r = (\epsilon_1, \dots, \epsilon_r) \in \mu_c^r, \qquad h \in \mathbb{N},$$
$$(\kappa_i)_{r-1} = (\kappa_1, \dots, \kappa_{r-1}) \in \mathbb{N}_0^{r-1}.$$

We define the *variant of cyclotomic multiple harmonic sums* as

$$\mathcal{S}_{(\kappa_i)_{r-1},h}((l_i)_r; (\epsilon_i)_r) = \sum_{\substack{(u_1,\dots,u_r)\in\mathbb{N}_0^r,\ u_1<h \\ \forall i\geq 2,\ u_{i-1}+\kappa_{i-1}h<u_i \\ \forall i\geq 2,\ u_i<u_{i-1}+(\kappa_{i-1}+1)h}} \left(\frac{\epsilon_2}{\epsilon_1}\right)^{u_1} \cdots \left(\frac{1}{\epsilon_r}\right)^{u_r} u_1^{l_1} \cdots u_r^{l_r} \in \mathbb{Q}(\mu_c),$$

where we formally let $0^0 = 1$.

Cyclotomic multiple Bernoulli numbers (CMBNs)

Definition

Let

$$c \in \mathbb{Z}_p^\times \cap \mathbb{N}_{\geq 2}, \qquad (l_i)_r = (l_1, \dots, l_r) \in \mathbb{N}_0^r,$$
$$(\epsilon_i)_r = (\epsilon_1, \dots, \epsilon_r) \in \mu_c^r, \qquad h \in \mathbb{N},$$
$$(\kappa_i)_{r-1} = (\kappa_1, \dots, \kappa_{r-1}) \in \mathbb{N}_0^{r-1}, \qquad \xi \in \mu_c,$$
$$l \in \{0, \dots, l_1 + \cdots + l_r + r\}.$$

We define the *variant of cyclotomic multiple Bernoulli numbers*

$$\mathcal{B}_{l,\xi}^{((l_i)_r;(\epsilon_i)_r;(\kappa_i)_{r-1})} \in \mathbb{Q}(\mu_c)$$

by

$$\mathcal{S}_{(\kappa_i)_{r-1},h}((l_i)_r;(\epsilon_i)_r) = \sum_{\substack{0 \leq l \leq l_1 + \cdots + l_r + r \\ \xi \in \mu_c}} \mathcal{B}_{l,\xi}^{((l_i)_r;(\epsilon_i)_r;(\kappa_i)_{r-1})} h^l \xi^h.$$

p-adic multiple L-functions

p-adic multiple L-function

Definition

Fix a prime number p, we put $\mathcal{O}_{\mathbb{C}_p}$ to be the ring of integers of $\mathbb{C}_p$. Let $\omega : \mathcal{O}_{\mathbb{C}_p}^\times \to \mathcal{O}_{\mathbb{C}_p}^\times$ be the Teichmüller character, and let $\langle x \rangle = \frac{x}{\omega(x)}$ for $x \in \mathcal{O}_{\mathbb{C}_p}^\times$. Set

$$\int_{\mathbb{Z}_p} f(x) d\mathfrak{m}_z(x) = \lim_{N \to \infty} \sum_{a=0}^{p^N-1} f(a) \mathfrak{m}_z(a + p^N \mathbb{Z}_p),$$

where $\mathfrak{m}_z$ is the measure defined by

$$\mathfrak{m}_z(j + p^N \mathbb{Z}_p) = \frac{z^j}{1 - z^{p^N}} \quad (0 \leq j \leq p^N - 1)$$

for $z \in \mathbb{P}^1(\mathbb{C}_p)$ with $|z-1|_p \geq 1$.

p-adic multiple L-function

Definition (continued)

Let p be a prime number and $(s_i)_r$ be the element in the set

$$\mathfrak{X}_r(d) := \{(s_1, \ldots, s_r) \in \mathbb{C}_p^r \mid |s_j|_p \leq d^{-1} p^{-\frac{1}{p-1}} \ (1 \leq j \leq r)\}.$$

Let $(k_i)_r \in \mathbb{Z}^r$, $c \in \mathbb{N}_{>1}$ which is prime to p, and

$$(\mathbb{Z}_p^r)' := \{(x_1, \ldots, x_r) \in \mathbb{Z}_p^r \mid p \nmid x_1, \ p \nmid (x_1 + x_2), \ \ldots, \ p \nmid (x_1 + \cdots + x_r)\}.$$

p-adic multiple L-function

Definition (continued)

The *p-adic multiple L-function* is defined by

$$L_{p,r}((s_i)_r;(\omega^{k_i})_r;(1)_r;c) := \int_{(\mathbb{Z}_p^r)'} \langle x_1\rangle^{-s_1}\langle x_1+x_2\rangle^{-s_2}\cdots\langle x_1+\cdots+x_r\rangle^{-s_r} \omega^{k_1}(x_1)\omega^{k_2}(x_1+x_2)\cdots\omega^{k_r}(x_1+\cdots+x_r)\prod_{i=1}^{r} d\tilde{\mathfrak{m}}_c(x_i),$$

where $\tilde{\mathfrak{m}}_c$ is the measure defined by

$$\tilde{\mathfrak{m}}_c := \sum_{\substack{\xi^c=1\\ \xi\neq 1}} \mathfrak{m}_\xi.$$

Expansion of special values of p-adic multiple L-function

Theorem

$$\left(p^{\sum_{i=1}^r n_i} L_{p,r}((n_i)_r;(\omega^{-n_i})_r;(1)_r;c)\right)_{p\in\mathcal{P}_c} =$$

$$\sum_{\mathrm{l}=(l_i)_r\in\mathbb{N}_0^r} \sum_{\substack{\epsilon\in\mu_c \\ (\epsilon_i)_r\in(\mu_c\setminus\{1\})^r}} \sum_{J=(P_1,P_2,P_3)\in E_r} \sum_{\substack{\xi\in\mu_c \\ \prod_{j\in P_3}\left(\frac{\epsilon_j^{(P_{2,3})}}{\epsilon_j}\right)^{-\kappa_j'}\xi=\epsilon}}$$

$$\frac{\epsilon}{\prod_{i=1}^r(1-\epsilon_i)} \prod_{i=1}^r \binom{-n_i}{l_i} \mathcal{B}_{0,\xi}^{(\mathrm{l}_J,\boldsymbol{\epsilon}_J,\boldsymbol{\kappa}_J)} \sum_{\delta\in\Delta(T_{r,J})} \mathfrak{H}(\mathrm{w}(\mathrm{l})_\delta)^{\mathrm{Frob}^{-1}}.$$

This infinite sum of series whose terms are $\mathbb{Q}(\mu_c)$-linear combinations of CMHVs with depth less than or equal to r and weight tending to infinity, and whose convergence holds for the topology on $\prod_{p\in\mathcal{P}_c}\mathbb{Q}_p(\mu_c)$ of the uniform convergence with respect to $p\in\mathcal{P}_c$.

Expansion of special values of p-adic multiple L-function

Theorem

$$\left(p^{\sum_{i=1}^r n_i} L_{p,r}((n_i)_r;(\omega^{-n_i})_r;(1)_r;c)\right)_{p\in\mathcal{P}_c} =$$

$$\sum_{\mathrm{l}=(l_i)_r\in\mathbb{N}_0^r} \sum_{\substack{\epsilon\in\mu_c \\ (\epsilon_i)_r\in(\mu_c\setminus\{1\})^r}} \sum_{J=(P_1,P_2,P_3)\in E_r} \frac{\epsilon}{\prod_{i=1}^r(1-\epsilon_i)} \prod_{i=1}^r \binom{-n_i}{l_i}$$

$$\sum_{(e_i)_{\sharp(P_2)}\in\{0\}\times\{0,1\}^{\sharp(P_2)-1}} \sum_{i_1=0}^{1-e_1}\cdots\sum_{i_{\sharp(P_2)}=0}^{1-e_{\sharp(P_2)}} (-1)^{\sharp(P_2)+\sum_{k=1}^{\sharp(P_2)}(i_k+l_{J,k})}$$

$$\zeta_{\sharp(P_2)-\sum_{k=1}^{\sharp(P_2)} e_k}\Big(\big(-l_{J,1}\circ_{e_2}\cdots\circ_{e_{\sharp(P_2)}} -l_{J,\sharp(P_2)}\big);$$

$$\big(\epsilon_{J,1}(1-e_1)\circ_{e_2}\cdots\circ_{e_{\sharp(P_2)}}\epsilon_{J,\sharp(P_2)}(1-e_{\sharp(P_2)})\big)\Big)$$

$$\frac{\delta}{\epsilon\prod_{j\in P_3}\left(\frac{\epsilon_j^{(P_{2,3})}}{\epsilon_j}\right)^{\kappa_j'}, \prod_{k=1}^{\sharp(P_2)}\epsilon_{J,k}^{-(\kappa_{J,k-1}+i_k)}} \sum_{\delta\in\Delta(T_{r,J})} \mathfrak{H}(\mathrm{w}(\mathrm{l})_\delta)^{\mathrm{Frob}^{-1}}.$$

The case of depth one

TBNs and CBNs

Example (The case $r = 1$)

For any root of unity ξ_1, TBNs is defined by

$$\mathcal{H}(t;\xi_1) = \frac{1}{1-\xi_1 e^t} = \sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi_1)\frac{t^n}{n!}.$$

Let $c \in \mathbb{Z}_p^\times \cap \mathbb{N}_{\geq 2}$. For any $l_1 \in \mathbb{N}_0$, $\epsilon_1 \in \mu_c$, and $h \in \mathbb{N}$, CHSs is defined by

$$\mathcal{S}_{\emptyset,h}(l_1;\epsilon_1) = \sum_{u_1=0}^{h-1} \epsilon_1^{-u_1} u_1^{l_1}.$$

For any $\xi \in \mu_c$ and $l \in \{0,\ldots,l_1+1\}$, CBNs is defined by

$$\mathcal{S}_{\emptyset,h}(l_1;\epsilon_1) = \sum_{\substack{l=0 \\ \xi\in\mu_c}}^{l_1+1} \mathcal{B}_{l,\xi}^{(l_1;\epsilon_1;\emptyset)} h^l \xi^h.$$

Key lemma

Lemma (The case $r = 1$)

Let $c \in \mathbb{Z}_p^\times \cap \mathbb{N}$, $c \geqslant 2$. Put $l_1 \in \mathbb{N}_0$ be a non-negative integer and $\epsilon_1 \in \mu_c$. Then, the following formula

$$\mathcal{B}_{l,\xi}^{(l_1;\epsilon_1;\emptyset)} = \mathfrak{B}_{l_1}(\epsilon_1^{-1})\delta_{\xi,1}\delta_{l,0} - \sum_{l'=0}^{l_1+1}\binom{l_1}{l'}\mathfrak{B}_{l_1-l'}(\epsilon_1^{-1})\delta_{\xi,\epsilon_1^{-1}}\delta_{l,l'}$$

holds for every ξ.

Proof.

Consider the following generating function with coefficient $\mathcal{S}_{\emptyset,h}(l_1;\epsilon_1)$

$$F(t) = \sum_{l_1=0}^{\infty}\mathcal{S}_{\emptyset,h}(l_1;\epsilon_1)\frac{t^{l_1}}{l_1!} = \sum_{l_1=0}^{\infty}\sum_{u_1=0}^{h-1}\epsilon_1^{-u_1}\frac{(u_1 t)^{l_1}}{l_1!}.$$

Key lemma

Proof (continued).

For convenience, we let $l_1 = l$, $u_1 = u$, and $\epsilon_1^{-1} = \xi$,

$$\begin{aligned} F(t) &= \sum_{l=0}^{\infty} \sum_{u=0}^{h-1} \xi^u \frac{(ut)^l}{l!} \\ &= \sum_{u=0}^{h-1} \xi^u \sum_{l=0}^{\infty} \frac{(ut)^l}{l!} \\ &= \sum_{u=0}^{h-1} \xi^u e^{ut} \\ &= \sum_{u=0}^{h-1} (\xi e^t)^u = \frac{1-(\xi e^t)^h}{1-\xi e^t}. \end{aligned}$$

Key lemma

Proof (continued).

By definition of TBNs, we obtain

$$
\begin{aligned}
F(t) &= \frac{1-(\xi e^t)^h}{1-\xi e^t} = (1-\xi^h e^{th})\frac{1}{1-\xi e^t} \\
&= \left(1-\xi^h \sum_{l=0}^{\infty} \frac{(th)^l}{l!}\right)\left(\sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi)\frac{t^n}{n!}\right) \\
&= \sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi)\frac{t^n}{n!} - \left(\xi^h \sum_{l=0}^{\infty} h^l \frac{t^l}{l!}\right)\left(\sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi)\frac{t^n}{n!}\right) \\
&= \sum_{m=-1}^{\infty} \mathfrak{B}_m(\xi)\frac{t^m}{m!} - \xi^h \sum_{m=-1}^{\infty} \sum_{l+n=m} h^l \mathfrak{B}_n(\xi)\frac{t^m}{l!n!}.
\end{aligned}
$$

Key lemma

Proof (continued).

$$\begin{aligned}
F(t) &= \sum_{m=-1}^{\infty} \mathfrak{B}_m(\xi)\frac{t^m}{m!} - \xi^h \sum_{m=-1}^{\infty} \sum_{l+n=m} h^l \mathfrak{B}_n(\xi)\frac{t^m}{l!n!} \\
&= \sum_{m=-1}^{\infty} \mathfrak{B}_m(\xi)\frac{t^m}{m!} - \xi^h \sum_{m=-1}^{\infty} \sum_{l=0}^{m+1} h^l \mathfrak{B}_{m-l}(\xi)\frac{t^m}{l!(m-l)!} \\
&= \sum_{m=-1}^{\infty} \mathfrak{B}_m(\xi)\frac{t^m}{m!} - \xi^h \sum_{m=-1}^{\infty} \sum_{l=0}^{m+1} h^l \mathfrak{B}_{m-l}(\xi)\binom{m}{l}\frac{t^m}{m!} \\
&= \sum_{m=-1}^{\infty} \left(\mathfrak{B}_m(\xi) - \xi^h \sum_{l=0}^{m+1} \binom{m}{l} h^l \mathfrak{B}_{m-l}(\xi)\right)\frac{t^m}{m!}.
\end{aligned}$$

Key lemma

Proof (continued).

By comparing the coefficients, we have

$$\mathcal{S}_{\emptyset,h}(l_1;\epsilon_1) = \mathfrak{B}_{l_1}(\epsilon_1^{-1}) - \sum_{l=0}^{l_1+1} \binom{l_1}{l} \mathfrak{B}_{l_1-l}(\epsilon_1^{-1})\epsilon_1^{-h} h^l.$$

By comparing the coefficients, we obtain

$$\mathcal{B}_{l,\xi}^{(l_1;\epsilon_1;\emptyset)} = \mathfrak{B}_{l_1}(\epsilon_1^{-1})\delta_{\xi,1}\delta_{l,0} - \sum_{l'=0}^{l_1+1} \binom{l_1}{l'} \mathfrak{B}_{l_1-l'}(\epsilon_1^{-1})\delta_{\xi,\epsilon_1^{-1}}\delta_{l,l'},$$

where δ is the Kronecker delta. □

The theorem of [FJ] and the theorem of [FKMT]

Theorem (The case $r = 1$)

$$\left(p^n L_p\left(n, \omega^{-n}, 1, c\right)\right)_{p\in\mathcal{P}_c} = \sum_{l=0}^{\infty} \binom{-n}{l} \sum_{\epsilon_1\in\mu_c\setminus\{1\}} \sum_{\xi\in\mu_c} \mathcal{B}_{0,\xi}^{(l,\epsilon_1^{-1},\emptyset)} \frac{\xi}{1-\epsilon_1} \mathfrak{H}\left(n+l, \epsilon_1^{-1}\right)^{Frob^{-1}}.$$

Theorem (The case $r = 1$)

$$\zeta(n;\xi;1) = (-1)^{n+1}\mathfrak{B}_n(\xi^{-1}).$$

Main theorem

Theorem (The case $r = 1$)

$$\left(p^n L_p\left(n, \omega^{-n}, 1, c\right)\right)_{p \in \mathcal{P}_c} = \sum_{l=0}^{\infty} \sum_{\epsilon \in \mu_c \setminus \{1\}} \binom{-n}{l} (-1)^{1+l} \zeta_1\left(-l; \epsilon^{-1}; 1\right) \mathfrak{H}(n+l, \epsilon^{-1})^{\mathrm{Frob}^{-1}}.$$

Proof.

By the theorems of [FJ] and [FKMT] and the key lemma, we obtain

$$\left(p^n L_p\left(n, \omega^{-n}, 1, c\right)\right)_{p \in \mathcal{P}_c} = \sum_{l=0}^{\infty} \binom{-n}{l} \sum_{\epsilon_1 \in \mu_c \setminus \{1\}} \sum_{\xi \in \mu_c} \frac{\xi}{1-\epsilon_1}$$
$$\left((-1)^{l+1} \zeta(l; \epsilon_1^{-1}; 1) \delta_{\xi,1} - (-1)^{l+1} \zeta(l; \epsilon_1^{-1}; 1) \delta_{\xi,\epsilon_1}\right) \mathfrak{H}\left(n+l, \epsilon_1^{-1}\right)^{\mathrm{Frob}^{-1}}.$$

□

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Thank you for your attention!