p-adic multiple zeta values of integer indices

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Admissible integer indices

Integer indices

Definition

For $r \in \mathbb{Z}_{\geq 1}$, an r-tuple $(k_1, \ldots, k_r)$ in $\mathbb{Z}^r$ is called an *integer index* and denoted by $\boldsymbol{k}$. For $r = 0$, the 0-tuple is an *integer index* defined as $\emptyset$.

Definition

The set of (resp. positive) integer indices $\mathcal{I}$ (resp. $\mathcal{I}_{>0}$) is defined by

$$\mathcal{I} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r \text{ (resp. } \mathcal{I}_{>0} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r_{>0}).$$

Definition

For two integer indices $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ and $\boldsymbol{k}' = (k'_1, \ldots, k'_{r'}) \in \mathbb{Z}^{r'}$, we define the *concatenation* of $\boldsymbol{k}, \boldsymbol{k}'$ to be $(k_1, \ldots, k_r, k'_1, \ldots, k'_{r'}) \in \mathbb{Z}^{r+r'}$ and denote it by $(\boldsymbol{k}, \boldsymbol{k}')$.

Integer indices

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define the *weight* of $\boldsymbol{k}$ to be $k_1 + \cdots + k_r$ and denote it by $\mathrm{wt}(\boldsymbol{k})$.

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define the *depth* of $\boldsymbol{k}$ to be r and denote it by $\mathrm{dep}(\boldsymbol{k})$.

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define the *tail index* $\boldsymbol{k}_t$ of $\boldsymbol{k}$ by $\boldsymbol{k}_t := (k_t, \ldots, k_r) \in \mathbb{Z}^{r-t+1}$ for $t = 1, \ldots, r$.

Regularizability index

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define the *regularizability index* $m_{\boldsymbol{k}}$ of $\boldsymbol{k}$ by

$$\begin{aligned} m_{\boldsymbol{k}} &:= \min \{ \mathrm{wt}(\boldsymbol{k}_t) - \mathrm{dep}(\boldsymbol{k}_t) \,|\, t = 1, \ldots, r \} \\ &= \min \left\{ \sum_{i=t}^{r} (k_i - 1) \,\middle|\, t = 1, \ldots, r \right\}. \end{aligned}$$

For $\boldsymbol{k} = \emptyset$, we define $m_\emptyset := \infty$.

Regularizability index of linear combinations

Definition

Let Σ be an element in the formal $\mathbb{Q}$-linear space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the support $\mathrm{Supp}(\Sigma)$ of Σ to be the set of integer indices satisfying

$$\Sigma = \sum_{\boldsymbol{k}\in\mathrm{Supp}(\Sigma)} c_{\boldsymbol{k}} \boldsymbol{k}$$

with $c_{\boldsymbol{k}} \in \mathbb{Q} \setminus \{0\}$.

Definition

Let Σ be an element in the formal $\mathbb{Q}$-linear space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the *regularizability index* m_Σ of Σ by

$$m_\Sigma := \min\left\{ m_{\boldsymbol{k}} \,\middle|\, \boldsymbol{k} \in \mathrm{Supp}(\Sigma) \right\}.$$

Admissible integer index

Definition

An integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ is called *admissible* if $m_{\boldsymbol{k}} > 0$.

Definition

The set of admissible (resp. positive) integer indices $\mathcal{I}^{\mathrm{adm}}$ (resp. $\mathcal{I}^{\mathrm{adm}}_{>0}$) is defined by

$$\mathcal{I}^{\mathrm{adm}} = \{\boldsymbol{k} \in \mathcal{I} | m_{\boldsymbol{k}} > 0\} \text{ (resp. } \mathcal{I}^{\mathrm{adm}}_{>0} = \{\boldsymbol{k} \in \mathcal{I}_{>0} | m_{\boldsymbol{k}} > 0\}).$$

Remark

For a positive integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, it is admissible if and only if $k_r > 1$.

Regularizable integer index

Definition

An integer index $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ is called *regularizable* if $m_{\boldsymbol{k}} \geq 0$.

Definition

The set of regularizable (resp. positive) integer indices $\mathcal{I}^{\mathrm{reg}}$ (resp. $\mathcal{I}^{\mathrm{reg}}_{>0}$) is defined by

$$\mathcal{I}^{\mathrm{reg}} = \{\boldsymbol{k} \in \mathcal{I} | m_{\boldsymbol{k}} \geq 0\} \text{ (resp. } \mathcal{I}^{\mathrm{reg}}_{>0} = \{\boldsymbol{k} \in \mathcal{I}_{>0} | m_{\boldsymbol{k}} \geq 0\}).$$

Remark

For a positive integer index $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, it is always regularizable.

Example

Example

Let $\boldsymbol{k} = (a, b)$ be an integer index. Then,

$$\mathrm{wt}(\boldsymbol{k}) = a + b,\ \mathrm{dep}(\boldsymbol{k}) = 2,$$

$$\boldsymbol{k}_1 = \boldsymbol{k} = (a, b),\ \boldsymbol{k}_2 = (b),$$

$$m_{\boldsymbol{k}} = \min\{a + b - 2, b - 1\}.$$

This implies $\boldsymbol{k}$ is admissible if $a + b - 2, b - 1 > 0$ and is regularizable if $a + b - 2, b - 1 \geq 0$.

Multiple polylogarithms

Multiple polylogarithms

Definition

Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ be an integer index. The *(single variable) multiple polylogarithms* are the power series defined by

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) := \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}},$$

where z is a complex number.

Definition

We consider the following $\mathbb{Q}$-linear spaces of multiple polylogarithms:

1. $\mathcal{MPL}^{\mathrm{adm}}_{>0} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}^{\mathrm{adm}}_{>0}\right\} \subset \mathbb{Q}[\![z]\!]$.
2. $\mathcal{MPL}_{>0} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}_{>0}(= \mathcal{I}^{\mathrm{reg}}_{>0})\right\} \subset \mathbb{Q}[\![z]\!]$.
3. $\mathcal{MPL}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}^{\mathrm{adm}}\right\} \subset \mathbb{Q}[\![z]\!]$.
4. $\mathcal{MPL}^{\mathrm{reg}} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}^{\mathrm{reg}}\right\} \subset \mathbb{Q}[\![z]\!]$.

Faulhaber's formula

We denote by B_i^+ (resp. B_i^-) be the Bernoulli numbers given by

$$\frac{t}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^+ \frac{t^n}{n!} \ \left(\text{resp. } \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n^- \frac{t^n}{n!}\right).$$

We note

$$B_1^+ = +\frac{1}{2},\ B_1^- = -\frac{1}{2},\ \text{and } B_n^- = B_n^+ \text{ for } n > 1.$$

Lemma (Faulhaber's formula)

Let k be a non-negative integer.

$$\sum_{n=1}^{m} n^k = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i} B_i^+ m^{k+1-i}$$

$$\left(\textit{resp. } \sum_{n=1}^{m-1} n^k = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i} B_i^- m^{k+1-i} - \delta_{k,0}\right).$$

Proposition for MPLs of integer index

Proposition (F.)

The following equalities hold:

1. $\mathcal{MPL}^{\mathrm{adm}}_{>0} = \mathcal{MPL}^{\mathrm{adm}}$.
2. $\mathcal{MPL}_{>0} (= \mathcal{MPL}^{\mathrm{reg}}_{>0}) = \mathcal{MPL}^{\mathrm{reg}}$.

Proof-Example.

Let $\boldsymbol{k} = (a, b)$ be an admissible integer index, i.e., $a + b - 2, b - 1 > 0$. We prove the multiple polylogarithm $\mathrm{Li}_{\boldsymbol{k}}(z) \in \mathcal{MPL}^{\mathrm{adm}}$ belongs to $\mathcal{MPL}^{\mathrm{adm}}_{>0}$. If $a > 0, b > 1$, then $\mathrm{Li}_{\boldsymbol{k}}(z) \in \mathcal{MPL}^{\mathrm{adm}}_{>0}$. If $a \leq 0, b > 1$, then by Faulhaber's formula we obtain

$$\mathrm{Li}_{\boldsymbol{k}}(z) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i^- \mathrm{Li}_{a+b-1+i}(z) - \delta_{-a,0} \mathrm{Li}_b(z) \in \mathcal{MPL}^{\mathrm{adm}}_{>0}.$$

One can inductively use Faulhaber's formula to prove this proposition. ☐

Positive-index map

Definition

The *positive-index map* is a $\mathbb{Q}$-linear map

$$\pi^+ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} \text{ (resp. } \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}_{>0}\})$$

defined by

$$\pi^+(\boldsymbol{k}) = \sum_{\boldsymbol{l}} c_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{l},$$

where the positive integer indices $\boldsymbol{l}$ and the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$ are uniquely obtained step by step through the inductive argument.

Example

Let $\boldsymbol{k} = (a, b)$ be an admissible integer index. If $a \leq 0, b > 1$, then

$$\pi^+(\boldsymbol{k}) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i^-(a+b-1+i) - \delta_{-a,0}(b).$$

Extension of the positive-index map

Remark

Actually, the positive-index map can be extended to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$, that is,

$$\pi^+ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, \qquad \pi^+(\boldsymbol{k}) = \sum_{\boldsymbol{l}} c_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{l},$$

where the integer indices

$$\boldsymbol{l} \in \{\emptyset\} \sqcup \bigsqcup_{r \in \mathbb{Z}_{>0}} \mathbb{Z}_{>0}^{r-1} \times \mathbb{Z}_{>m_{\boldsymbol{k}}}$$

and the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$ are obtained step by step through the same inductive argument. For integer indices $\boldsymbol{k} \in \mathcal{I}$, we have the identity

$$\mathrm{Li}_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} \, \mathrm{Li}_{\boldsymbol{l}}(z) \in \mathbb{Q}[\![z]\!].$$

Shuffle product

Correspondence of integer indices and words

We recall the shuffle product for integer indices introduced by Ebrahimi-Fard, Manchon, and Singer.

Definition

Let X be the alphabet $\{j, d, y\}$, and let W denote the set of words on the alphabet X, subject to the rule $jd = dj = \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define

$$w_{\boldsymbol{k}} = j^{k_r} y \cdots j^{k_1} y$$

to be the *corresponding word*. Conversely for $w = j^{k_r} y \cdots j^{k_1} y$, we define

$$\boldsymbol{k}_w = (k_1, \ldots, k_r)$$

to be the *corresponding integer index*.

Remark

We have a one-to-one correspondence between the set Wy and the set $\mathcal{I}$.

Shuffle product for integer indices

Definition ([Ebrahimi-Fard, Manchon, Singer])

We define the shuffle product by $\mathbf{1} \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, \mathbf{1} = w$ for any word $w \in Wy$, and recursively with respect to the sum of the lengths of two words in Wy:

1. $yu \sqcup\!\sqcup\, v := u \sqcup\!\sqcup\, yv := y(u \sqcup\!\sqcup\, v)$,
2. $ju \sqcup\!\sqcup\, jv := j(u \sqcup\!\sqcup\, jv) + j(ju \sqcup\!\sqcup\, v)$,
3. $du \sqcup\!\sqcup\, dv := d(u \sqcup\!\sqcup\, dv) - u \sqcup\!\sqcup\, d^2v$,
4. $du \sqcup\!\sqcup\, jv := d(u \sqcup\!\sqcup\, jv) - u \sqcup\!\sqcup\, v$,
5. $ju \sqcup\!\sqcup\, dv := d(ju \sqcup\!\sqcup\, v) - u \sqcup\!\sqcup\, v$.

Definition

Let $\boldsymbol{k}, \boldsymbol{k}' \in \mathcal{I}$ be two integer indices. We define the shuffle product for integer indices by

$$\boldsymbol{k} \sqcup\!\sqcup\, \boldsymbol{k}' := \boldsymbol{k}_{w_{\boldsymbol{k}} \sqcup\!\sqcup\, w_{\boldsymbol{k}'}}.$$

Example of shuffle product

Example

The corresponding words of $(1,-1),(-1,1)$ are $dyjy, jydy$, and

$$\begin{aligned}
& dyjy \shuffle jydy & \\
=& d(yjy \shuffle jydy) - yjy \shuffle ydy & \text{by } 4\\
=& dy(jy \shuffle jydy) - yy(jy \shuffle dy) & \text{by } 1\\
=& dy(j(y \shuffle jydy) + j(jy \shuffle ydy)) - yy(jy \shuffle dy) & \text{by } 2\\
=& dy(jyjydy + jy(jy \shuffle dy)) - yy(jy \shuffle dy) & \text{by } 1\\
=& dyjyjydy + dyjy(d(jy \shuffle y) - y \shuffle y) - yy(d(jy \shuffle y) - y \shuffle y) & \text{by } 5\\
=& dyjyjydy + dyjydyjy - dyjyyy - yydyjy + yyyy. & \text{by } 1
\end{aligned}$$

Hence, the shuffle product $(1,-1) \shuffle (-1,1)$ of them is

$$(-1,1,1,-1)+(1,-1,1,-1)-(0,0,1,-1)-(1,-1,0,0)+(0,0,0,0).$$

Regularizability index for shuffle product

Proposition (F.)

Let $\boldsymbol{k}, \boldsymbol{k}'$ *be two integer indices. Then,*

$$m_{\boldsymbol{k} \sqcup\!\sqcup \boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\}.$$

Proof sketch.

By induction on the sum of the lengths of $w_{\boldsymbol{k}}$ and $w_{\boldsymbol{k}'}$, we check all recursion conditions in the definition of the shuffle product by considering cases according to the comparison between the index and 0. ☐

Example

$$\begin{aligned} m_{(1,-1) \sqcup\!\sqcup (-1,1)} &= m_{(-1,1,1,-1)+(1,-1,1,-1)-(0,0,1,-1)-(1,-1,0,0)+(0,0,0,0)} \\ &= \min\{-4,-4,-4,-4,-4\} = -4 \end{aligned}$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2,-2,-4\} = -4$$

Regularizability index for shuffle product

Corollary (F.)

The space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}$ *(resp.* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\}$*) of admissible (resp. regularizable) integer indices is a* $\mathbb{Q}$*-subalgebra of* $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, ш)$.

Proof.

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\boldsymbol{k} ш \boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, ш)$. □

Positive-index map and shuffle product

Theorem (F.)

Let
$\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, ⧢), \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, ⧢)$
be the $\mathbb{Q}$-algebra with shuffle product ⧢. Then, the positive-index map

$$\pi^{+} : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}$$

is a $\mathbb{Q}$-algebra homomorphism, that is,

$$⧢ \circ (\pi^{+} \otimes \pi^{+}) = \pi^{+} \circ ⧢.$$

Proof sketch.

We show the general formula below by induction on the sum of the lengths of $w_{\boldsymbol{k}}$ and $w_{\boldsymbol{k}'}$.

$$\pi^{+} \circ ⧢ \circ (\pi^{+} \otimes \pi^{+}) = \pi^{+} \circ ⧢ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}.$$

□

Stuffle product

Stuffle product for integer indices

We recall the stuffle product for integer indices.

Definition

We define the stuffle product

$$* : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$$

recursively by:

$$\emptyset * \boldsymbol{k} = \boldsymbol{k} * \emptyset := \boldsymbol{k}$$

where $\boldsymbol{k} \in \mathcal{I}$ is an integer index, and for two integer indices $(\boldsymbol{k}, k), (\boldsymbol{k}', k')$ with $k, k' \in \mathbb{Z}$ and $\boldsymbol{k}, \boldsymbol{k}' \in \mathcal{I}$,

$$(\boldsymbol{k}, k) * (\boldsymbol{k}', k') := (\boldsymbol{k} * (\boldsymbol{k}', k'), k) + ((\boldsymbol{k}, k) * \boldsymbol{k}', k') + (\boldsymbol{k} * \boldsymbol{k}', k + k').$$

Example of stuffle product

Example

The stuffle product of two integer indices $(1,-1)$ and $(-1,1)$ is

$$\begin{aligned}
&(1,-1)*(-1,1)\\
=&((1)*(-1,1),-1)+((1,-1)*(-1),1)+((1)*(-1),-1+1)\\
=&((-1,1,1)+((1)*(-1),1)+(-1,2),-1)\\
&+(((1)*(-1),-1)+(1,-1,-1)+(1,-2),1)\\
&+((1,-1)+(-1,1)+(0),0)\\
=&((-1,1,1)+((1,-1)+(-1,1)+(0),1)+(-1,2),-1)\\
&+(((1,-1)+(-1,1)+(0),-1)+(1,-1,-1)+(1,-2),1)\\
&+(1,-1,0)+(-1,1,0)+(0,0)\\
=&(-1,1,1,-1)+(1,-1,1,-1)+(-1,1,1,-1)+(0,1,-1)\\
&+(-1,2,-1)+(1,-1,-1,1)+(-1,1,-1,1)+(1,-1,-1,1)\\
&+(0,-1,1)+(1,-2,1)+(1,-1,0)+(-1,1,0)+(0,0).
\end{aligned}$$

Regularizability index for stuffle product

Proposition (F.)

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two integer indices. Then,

$$m_{\boldsymbol{k}*\boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\}.$$

Proof sketch.

By induction on the sum of the depth of $\boldsymbol{k}$ and $\boldsymbol{k}'$, we check the recursion condition in the definition of the stuffle product. ☐

Example

$$m_{(1,-1)*(-1,1)}$$
$$= \min\{-4, -4, -4, -3, -3, -4, -4, -4, -3, -3, -3, -3, -2\} = -4$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2, -2, -4\} = -4$$

Regularizability index for stuffle product

Corollary (F.)

The space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}$ *(resp.* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\}$*) of admissible (resp. regularizable) integer indices is a* $\mathbb{Q}$*-subalgebra of* $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$*.*

Proof.

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\boldsymbol{k}*\boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$. ☐

Positive-index map and stuffle product

Theorem (F.)

Let

$\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, *), \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, *)$ *be the* $\mathbb{Q}$*-algebra with stuffle product* $*$*. Then, the positive-index map*

$$\pi^{+} : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}$$

is a $\mathbb{Q}$*-algebra homomorphism, that is,*

$$* \circ (\pi^{+} \otimes \pi^{+}) = \pi^{+} \circ *.$$

Proof sketch.

By induction on the sum of the depth of $\boldsymbol{k}$ and $\boldsymbol{k}'$, we check the recursion condition in the definition of the stuffle product. ☐

p-adic multiple zeta values

p-adic (single variable) multiple polylogarithms

Definition

Fix a prime p. For an admissible positive integer index $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$, the *$p$-adic (single variable) multiple polylogarithms* are defined by the following series

$$\mathrm{Li}^p_{k_1,\dots,k_r}(z) = \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} \in \mathbb{Q}_p[\![z]\!].$$

Remark

Let $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. Let π^+ be the positive-index map. Then,

$$\mathrm{Li}^p_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{l}\in\mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}}\,\mathrm{Li}^p_{\boldsymbol{l}}(z)$$

holds for the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$.

p-adic multiple zeta values

Definition ([Furusho])

Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ be an admissible positive integer index. The *p-adic multiple zeta values* are defined by the following specific limit

$$\zeta_p(k_1, \ldots, k_r) := \lim_{z \to 1}{}' \mathrm{Li}^p_{k_1,\ldots,k_r}(z) \in \mathbb{Q}_p.$$

Definition

Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. We define the *p-adic multiple zeta value* $\zeta_p(k_1, \ldots, k_r)$ for an integer index by

$$\zeta_p(k_1, \ldots, k_r) := \lim_{z \to 1}{}' \mathrm{Li}^p_{\boldsymbol{k}}(z)$$

$$= \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} \lim_{z \to 1}{}' \mathrm{Li}^p_{\boldsymbol{l}}(z) = \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} \zeta_p(\boldsymbol{l}) \in \mathbb{Q}_p,$$

where π^+ is the positive-index map.

Double shuffle for p-adic MZVs of positive integer indices

We consider the $\mathbb{Q}$-linear map

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

defined by $\boldsymbol{k} \mapsto \zeta_p(\boldsymbol{k})$.

Remark ([Besser, Furusho], [Furusho, Jafari])

The p-adic multiple zeta values for admissible positive integer indices satisfy the double shuffle relation, that is, the $\mathbb{Q}$-linear map ζ_p is the $\mathbb{Q}$-algebra homomorphism with respect to the shuffle product $\sqcup\!\sqcup$

$$\zeta_p : (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, \sqcup\!\sqcup) \to \mathbb{Q}_p$$

and is the $\mathbb{Q}$-algebra homomorphism with respect to the stuffle product $*$

$$\zeta_p : (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, *) \to \mathbb{Q}_p.$$

Double shuffle for p-adic MZVs of integer indices

Proposition (F.)

Let $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, ш)$ *be the* $\mathbb{Q}$*-algebra of admissible integer indices. Then, the linear map*

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

is a $\mathbb{Q}$*-algebra homomorphism, i.e.,* $\zeta_p(\boldsymbol{k} \,ш\, \boldsymbol{k}') = \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}')$.

Proposition (F.)

Let $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, *)$ *be the* $\mathbb{Q}$*-algebra of admissible integer indices. Then, the linear map*

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

is a $\mathbb{Q}$*-algebra homomorphism, i.e.,* $\zeta_p(\boldsymbol{k} * \boldsymbol{k}') = \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}')$.

Double shuffle for p-adic MZVs of integer indices

Proof.

The composition of algebra homomorphism ζ_p and π^+ is still an algebra homomorphism. ☐

Theorem (F.)

Let $\boldsymbol{k}, \boldsymbol{k}'$ be admissible integer indices. The p-adic multiple zeta values at integer indices satisfy the double shuffle relation, that is,

$$\zeta_p(\boldsymbol{k} \sqcup\!\sqcup \boldsymbol{k}') = \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}') = \zeta_p(\boldsymbol{k} * \boldsymbol{k}').$$

Proof.

By the two propositions above, one can see that

$$\zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}') = \zeta_p(\boldsymbol{k} \sqcup\!\sqcup \boldsymbol{k}') \text{ and } \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}') = \zeta_p(\boldsymbol{k} * \boldsymbol{k}').$$

☐

Example

Example

For admissible integer indices (a) and (b, c) with $b < 0$, we have

$$\zeta_p((a) \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup (b,c)) = \sum_{i=0}^{c} \binom{a+i}{i} \zeta_p(b, c-i, a+i) +$$

$$\sum_{i=0}^{a} \binom{c+i}{i} \left[\sum_{j=0}^{\min\{a-i-1,-b\}} (-1)^j \binom{-b}{j} \zeta_p(a-i-j, b+j, c+i) \right.$$

$$\left. +(-1)^{a-i} \sum_{j=0}^{-b-a+i} \binom{-b-1-j}{a-i-1} \zeta_p(-j, b+a-i+j, c+i) \right],$$

where we formally let $\binom{n}{-1} = \delta_{n,-1}$ and

$$\zeta_p((a) * (b,c))$$
$$= \zeta_p(b,c,a) + \zeta_p(b,a,c) + \zeta_p(a,b,c) + \zeta_p(a+b,c) + \zeta_p(b,a+c).$$

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Thank you for your attention!