

p -adic multiple zeta values and double shuffle relations for integer indices

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Multiple zeta values

Definition

For $r \in \mathbb{N}$, an r -tuple (k_1, \dots, k_r) in \mathbb{N}^r is called a **positive integer index** and denoted by \mathbf{k} . For $r = 0$, the 0-tuple is a **positive integer index** defined as \emptyset .

Definition

The positive integer indices (k_1, \dots, k_r) is called **admissible** if $k_r > 1$.

Definition

For admissible positive integer indices (k_1, \dots, k_r) , **multiple zeta values** are the real numbers defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}.$$

Multiple zeta values

Remark

When $\mathbf{k} = (k)$, multiple zeta values $\zeta(k)$ are special values of the Riemann zeta function.

Remark

For positive integer indices (k_1, \dots, k_r) with $k_r = 1$, the series

$$\sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

will diverge. Although it diverges, one can define multiple zeta values through regularization and prove that it is compatible with double shuffle relations.

How about if the positive integer indices of this series are integer indices? Will it diverge or converge?

Integer indices

Definition

For $r \in \mathbb{N}$, an r -tuple (k_1, \dots, k_r) in \mathbb{Z}^r is called an **integer index** and denoted by \mathbf{k} . For $r = 0$, the 0-tuple is an **integer index** defined as \emptyset .

Definition

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$,

- the **weight** of \mathbf{k} is defined by $\text{wt}(\mathbf{k}) := k_1 + \dots + k_r$,
- the **depth** of \mathbf{k} is defined by $\text{dep}(\mathbf{k}) := r$,
- the **tail index** of \mathbf{k} is defined by $\mathbf{k}_t := (k_t, \dots, k_r) \in \mathbb{Z}^{r-t+1}$, where $t = 1, \dots, r$.

Definition

For two integer indices $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $\mathbf{k}' = (k'_1, \dots, k'_{r'}) \in \mathbb{Z}^{r'}$, we define the **concatenation** of \mathbf{k}, \mathbf{k}' to be $(k_1, \dots, k_r, k'_1, \dots, k'_{r'}) \in \mathbb{Z}^{r+r'}$ and denote it by $(\mathbf{k}, \mathbf{k}')$.

Integer indices

Definition

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we define the **regularizability index** $m_{\mathbf{k}}$ of \mathbf{k} by

$$\begin{aligned} m_{\mathbf{k}} &:= \min \{ \text{wt}(\mathbf{k}_t) - \text{dep}(\mathbf{k}_t) \mid t = 1, \dots, r \} \\ &= \min \left\{ \sum_{i=t}^r (k_i - 1) \mid t = 1, \dots, r \right\}. \end{aligned}$$

For $\mathbf{k} = \emptyset$, we define $m_{\emptyset} := \infty$.

Admissible and regularizable

Definition

An integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ is called **admissible** if $m_{\mathbf{k}} > 0$, is called **regularizable** if $m_{\mathbf{k}} \geq 0$.

Remark

For a positive integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, it is admissible if and only if $k_r > 1$, and it is always regularizable.

Proposition (F.)

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, it is admissible if and only if the series

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

converge, and it is regularizable if and only if one can define multiple zeta values through regularization.

Example

Example

Let $\mathbf{k} = (a, b)$ be an integer index. Then,

$$\text{wt}(\mathbf{k}) = a + b, \quad \text{dep}(\mathbf{k}) = 2,$$

$$\mathbf{k}_1 = \mathbf{k} = (a, b), \quad \mathbf{k}_2 = (b),$$

$$m_{\mathbf{k}} = \min\{a + b - 2, b - 1\}.$$

This implies \mathbf{k} is admissible if $a + b - 2, b - 1 > 0$ and is regularizable if $a + b - 2, b - 1 \geq 0$.

Admissible and regularizable

Definition

The set of (resp. positive) integer indices \mathcal{I} (resp. $\mathcal{I}_{>0}$) is defined by

$$\mathcal{I} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r \quad (\text{resp. } \mathcal{I}_{>0} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{>0}^r).$$

Definition

The set of admissible (resp. positive) integer indices \mathcal{I}^{adm} (resp. $\mathcal{I}_{>0}^{\text{adm}}$) is defined by

$$\mathcal{I}^{\text{adm}} = \{\mathbf{k} \in \mathcal{I} \mid m_{\mathbf{k}} > 0\} \quad (\text{resp. } \mathcal{I}_{>0}^{\text{adm}} = \{\mathbf{k} \in \mathcal{I}_{>0} \mid m_{\mathbf{k}} > 0\}).$$

The set of regularizable (resp. positive) integer indices \mathcal{I}^{reg} (resp. $\mathcal{I}_{>0}^{\text{reg}}$) is defined by

$$\mathcal{I}^{\text{reg}} = \{\mathbf{k} \in \mathcal{I} \mid m_{\mathbf{k}} \geq 0\} \quad (\text{resp. } \mathcal{I}_{>0}^{\text{reg}} = \{\mathbf{k} \in \mathcal{I}_{>0} \mid m_{\mathbf{k}} \geq 0\}).$$

Admissible and regularizable

Definition

Let Σ be an element in the formal \mathbb{Q} -linear space $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **support** $\text{Supp}(\Sigma)$ of Σ to be the set of integer indices satisfying

$$\Sigma = \sum_{\mathbf{k} \in \text{Supp}(\Sigma)} c_{\mathbf{k}} \mathbf{k}$$

with $c_{\mathbf{k}} \in \mathbb{Q} \setminus \{0\}$.

Definition

Let Σ be an element in the formal \mathbb{Q} -linear space $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **regularizability index** m_{Σ} of Σ by

$$m_{\Sigma} := \min \{m_{\mathbf{k}} \mid \mathbf{k} \in \text{Supp}(\Sigma)\}.$$

Multiple polylogarithms

Definition

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an integer index. The **(single variable) multiple polylogarithms** are the power series defined by

$$\mathrm{Li}_{k_1, \dots, k_r}(z) := \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}},$$

where z is a complex number.

Definition

We consider the following \mathbb{Q} -linear spaces of multiple polylogarithms:

- ① $\mathcal{MPL}_{>0}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}_{>0}^{\mathrm{adm}} \} \subset \mathbb{Q}[z].$
- ② $\mathcal{MPL}_{>0} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}_{>0} (= \mathcal{I}_{>0}^{\mathrm{reg}}) \} \subset \mathbb{Q}[z].$
- ③ $\mathcal{MPL}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}^{\mathrm{adm}} \} \subset \mathbb{Q}[z].$
- ④ $\mathcal{MPL}^{\mathrm{reg}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}^{\mathrm{reg}} \} \subset \mathbb{Q}[z].$

The proposition for MPLs

Proposition (F.)

The following equalities hold:

- 1 $MPL_{>0}^{\text{adm}} = MPL^{\text{adm}}$.
- 2 $MPL_{>0}(= MPL_{>0}^{\text{reg}}) = MPL^{\text{reg}}$.

Faulhaber's formula

We denote by B_i be the Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note $B_1 = -\frac{1}{2}$.

Lemma (Faulhaber's formula)

Let k be a non-negative integer.

$$\sum_{n=1}^{m-1} n^k = \frac{1}{k+1} \sum_{i=0}^k \binom{k+1}{i} B_i m^{k+1-i} - \delta_{k,0}.$$

Proof of the proposition

Proof-Example.

Let $\mathbf{k} = (a, b)$ be an admissible integer index, i.e., $a + b - 2, b - 1 > 0$. We prove the multiple polylogarithm $\text{Li}_{\mathbf{k}}(z) \in \mathcal{MPL}^{\text{adm}}$ belongs to $\mathcal{MPL}_{>0}^{\text{adm}}$. If $a > 0, b > 1$, then $\text{Li}_{\mathbf{k}}(z) \in \mathcal{MPL}_{>0}^{\text{adm}}$. If $a \leq 0, b > 1$, then by Faulhaber's formula we obtain

$$\text{Li}_{\mathbf{k}}(z) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i \text{Li}_{a+b-1+i}(z) - \delta_{-a,0} \text{Li}_b(z) \in \mathcal{MPL}_{>0}^{\text{adm}}.$$

One can inductively use Faulhaber's formula to prove this proposition. □

Positive-index map

Definition

The **positive-index map** is a \mathbb{Q} -linear map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} \quad (\text{resp. } \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{reg}}\})$$

defined by

$$\pi^+(\mathbf{k}) = \sum_I c_{\mathbf{k},I} I,$$

where the positive integer indices I and the coefficients $c_{\mathbf{k},I} \in \mathbb{Q} \setminus \{0\}$ are uniquely obtained step by step through the inductive argument.

Example

Let $\mathbf{k} = (a, b)$ be an admissible integer index. If $a \leq 0, b > 1$, then

$$\pi^+(\mathbf{k}) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i(a+b-1+i) - \delta_{-a,0}(b).$$

Positive-index map

Remark

Actually, the positive-index map can be extended to $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$, that is,

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \quad \pi^+(\mathbf{k}) = \sum_I c_{\mathbf{k},I} I,$$

where the integer indices

$$I \in \{\emptyset\} \sqcup \bigsqcup_{r \in \mathbb{Z}_{>0}} \mathbb{Z}_{>0}^{r-1} \times \mathbb{Z}_{>0}^{m_k}$$

and the coefficients $c_{\mathbf{k},I} \in \mathbb{Q} \setminus \{0\}$ are obtained step by step through the same inductive argument. For integer indices $\mathbf{k} \in \mathcal{I}$, we have the identity

$$\text{Li}_{\mathbf{k}}(z) = \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k},I} \text{Li}_I(z) \in \mathbb{Q}[[z]].$$

Shuffle product

Shuffle product and MZVs

Definition

Let $a_1, \dots, a_k \in \{0, 1\}$ with $a_1 = 1, a_k = 0$. The **iterated integral** is defined by

$$I(a_1, \dots, a_k) = \int_{0 < t_1 < \dots < t_k < 1} \prod_{j=1}^k \frac{(-1)^{a_j} dt_j}{t_j - a_j}.$$

Remark

Multiple zeta values admit the iterated integral expression

$$\zeta(k_1, \dots, k_d) = I(1, \underbrace{0, \dots, 0}_{k_1-1 \text{ times}}, \dots, 1, \underbrace{0, \dots, 0}_{k_d-1 \text{ times}}).$$

Example

$$\zeta(4) = I(1, 0, 0, 0), \quad \zeta(2, 3) = I(1, 0, 1, 0, 0).$$

Shuffle product and MZVs

Example

Using the iterated integral expression

$$\zeta(k_1, \dots, k_d) = I(0; 1, \{0\}^{k_1-1}, \dots, 1, \{0\}^{k_d-1}; 1)$$

to calculate

$$\begin{aligned} \zeta(2) \times \zeta(2) &= I(1, 0) \times I(1, 0) \\ &= \int_{0 < x_1 < x_2 < 1} \frac{dx_1}{x_1-1} \frac{dx_2}{x_2} \times \int_{0 < y_1 < y_2 < 1} \frac{dy_1}{y_1-1} \frac{dy_2}{y_2} \\ &= \int_{0 < x_1 < x_2 < y_1 < y_2 < 1} \frac{dx_1}{x_1-1} \dots \frac{dy_2}{y_2} + \dots + \int_{0 < y_1 < y_2 < x_1 < x_2 < 1} \frac{dy_1}{y_1-1} \dots \frac{dx_2}{x_2} \\ &= I(1, 0, 1, 0) + I(1, 1, 0, 0) + I(1, 1, 0, 0) + I(1, 1, 0, 0) \\ &\quad + I(1, 1, 0, 0) + I(1, 0, 1, 0) \\ &= 2\zeta(2, 2) + 4\zeta(1, 3) \end{aligned}$$

Shuffle product for integer indices

We recall the shuffle product for integer indices introduced by Ebrahimi-Fard, Manchon, and Singer.

Definition

Let X be the alphabet $\{j, d, y\}$, and let W denote the set of words on the alphabet X , subject to the rule $jd = dj = \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we define

$$w_{\mathbf{k}} = j^{k_r} y \cdots j^{k_1} y$$

to be the **corresponding word**. Conversely for $w = j^{k_r} y \cdots j^{k_1} y$, we define

$$\mathbf{k}_w = (k_1, \dots, k_r)$$

to be the **corresponding integer index**.

Remark

We have a one-to-one correspondence between the set W_y and the set \mathcal{I} .

Shuffle product for integer indices

Definition ([Ebrahimi-Fard, Manchon, Singer])

We define the shuffle product by $\mathbf{1} \sqcup w = w \sqcup \mathbf{1} = w$ for any word $w \in \mathcal{W}_y$, and recursively with respect to the sum of the lengths of two words in \mathcal{W}_y :

- ① $yu \sqcup v := u \sqcup yv := y(u \sqcup v),$
- ② $ju \sqcup jv := j(u \sqcup jv) + j(ju \sqcup v),$
- ③ $du \sqcup dv := d(u \sqcup dv) - u \sqcup d^2v,$
- ④ $du \sqcup jv := d(u \sqcup jv) - u \sqcup v,$
- ⑤ $ju \sqcup dv := d(ju \sqcup v) - u \sqcup v.$

Definition

Let $\mathbf{k}, \mathbf{k}' \in \mathcal{I}$ be two integer indices. We define the shuffle product for integer indices by

$$\mathbf{k} \sqcup \mathbf{k}' := \mathbf{k}_{w_{\mathbf{k}} \sqcup w_{\mathbf{k}'}}.$$

Shuffle product for integer indices

Example

The corresponding words of $(1, -1)$, $(-1, 1)$ are $dyjy$, $jydy$, and

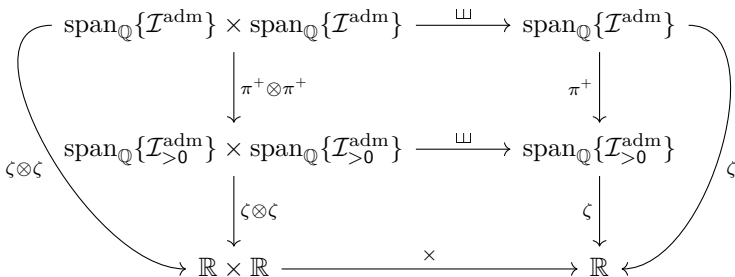
$$\begin{aligned}
 & dyjy \sqcup jydy \\
 = & d(yjy \sqcup jydy) - yjy \sqcup ydy && \text{by 4} \\
 = & dy(jy \sqcup jydy) - yy(jy \sqcup dy) && \text{by 1} \\
 = & dy(j(y \sqcup jydy) + j(jy \sqcup ydy)) - yy(jy \sqcup dy) && \text{by 2} \\
 = & dy(jyjydy + jy(jy \sqcup dy)) - yy(jy \sqcup dy) && \text{by 1} \\
 = & dyjyjydy + dyjy(d(jy \sqcup y) - y \sqcup y) - yy(d(jy \sqcup y) - y \sqcup y) && \text{by 5} \\
 = & dyjyjydy + dyjydyjy - dyjyyy - yydyjy + yyyy. && \text{by 1}
 \end{aligned}$$

Hence, the shuffle product $(1, -1) \sqcup (-1, 1)$ of them is

$$(-1, 1, 1, -1) + (1, -1, 1, -1) - (0, 0, 1, -1) - (1, -1, 0, 0) + (0, 0, 0, 0).$$

Positive-index map and shuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the shuffle algebra?

Positive-index map and shuffle product

Proposition (F.)

Let \mathbf{k}, \mathbf{k}' be two integer indices. Then,

$$m_{\mathbf{k} \sqcup \mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\}.$$

Proof sketch.

By induction on the sum of the lengths of $w_{\mathbf{k}}$ and $w_{\mathbf{k}'}$, we check all recursion conditions in the definition of the shuffle product by considering cases according to the comparison between the index and 0. \square

Example

$$\begin{aligned} m_{(1,-1) \sqcup (-1,1)} &= m_{(-1,1,1,-1) + (1,-1,1,-1) - (0,0,1,-1) - (1,-1,0,0) + (0,0,0,0)} \\ &= \min\{-4, -4, -4, -4, -4\} = -4 \end{aligned}$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2, -2, -4\} = -4$$

Positive-index map and shuffle product

Corollary (F.)

The space $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}$ (resp. $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\}$) of admissible (resp. regularizable) integer indices is a \mathbb{Q} -subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \sqcup)$.

Proof.

Let \mathbf{k}, \mathbf{k}' be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\mathbf{k} \sqcup \mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \sqcup)$. □

Positive-index map and shuffle product

Theorem (F.)

Let

$\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, \sqcup), \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, \sqcup)$
be the \mathbb{Q} -algebra with shuffle product \sqcup . Then, the positive-index map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}$$

is a \mathbb{Q} -algebra homomorphism, that is,

$$\sqcup \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ \sqcup.$$

Proof sketch.

We show the general formula below by induction on the sum of the lengths of w_k and $w_{k'}$.

$$\pi^+ \circ \sqcup \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ \sqcup : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}. \quad \square$$

Shuffle product and MZVs

Example

Using the series expression

$$\zeta(k_1, \dots, k_d) = \sum_{0 < n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \dots n_d^{k_d}}$$

to calculate

$$\begin{aligned} \zeta(2) \times \zeta(2) &= \sum_{0 < n} \frac{1}{n^2} \times \sum_{0 < m} \frac{1}{m^2} \\ &= \sum_{0 < n < m} \frac{1}{n^2 m^2} + \sum_{0 < n = m} \frac{1}{n^2 m^2} + \sum_{0 < m < n} \frac{1}{m^2 n^2} \\ &= \zeta(2, 2) + \zeta(4) + \zeta(2, 2). \end{aligned}$$

Double shuffle relation:

$$2\zeta(2, 2) + \zeta(4) = \zeta(2) \times \zeta(2) = 2\zeta(2, 2) + 4\zeta(1, 3).$$

Shuffle product for integer indices

We recall the shuffle product for integer indices.

Definition

We define the shuffle product

$$* : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}$$

recursively by:

$$\emptyset * \mathbf{k} = \mathbf{k} * \emptyset := \mathbf{k}$$

where $\mathbf{k} \in \mathcal{I}$ is an integer index, and for two integer indices $(\mathbf{k}, k), (\mathbf{k}', k')$ with $k, k' \in \mathbb{Z}$ and $\mathbf{k}, \mathbf{k}' \in \mathcal{I}$,

$$(\mathbf{k}, k) * (\mathbf{k}', k') := (\mathbf{k} * (\mathbf{k}', k'), k) + ((\mathbf{k}, k) * \mathbf{k}', k') + (\mathbf{k} * \mathbf{k}', k + k').$$

Shuffle product for integer indices

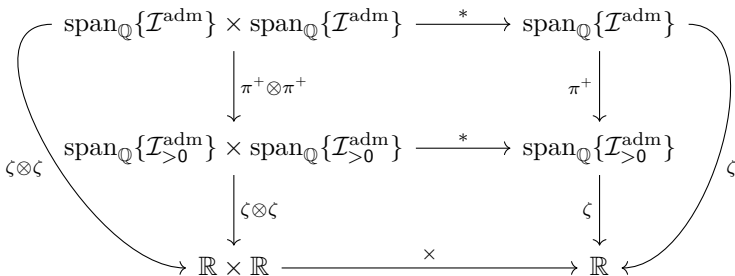
Example

The shuffle product of two integer indices $(1, -1)$ and $(-1, 1)$ is

$$\begin{aligned}
 & (1, -1) * (-1, 1) \\
 = & ((1) * (-1, 1), -1) + ((1, -1) * (-1), 1) + ((1) * (-1), -1 + 1) \\
 = & ((-1, 1, 1) + ((1) * (-1), 1) + (-1, 2), -1) \\
 & + (((1) * (-1), -1) + (1, -1, -1) + (1, -2), 1) \\
 & + ((1, -1) + (-1, 1) + (0), 0) \\
 = & ((-1, 1, 1) + ((1, -1) + (-1, 1) + (0), 1) + (-1, 2), -1) \\
 & + (((1, -1) + (-1, 1) + (0), -1) + (1, -1, -1) + (1, -2), 1) \\
 & + (1, -1, 0) + (-1, 1, 0) + (0, 0) \\
 = & (-1, 1, 1, -1) + (1, -1, 1, -1) + (-1, 1, 1, -1) + (0, 1, -1) \\
 & + (-1, 2, -1) + (1, -1, -1, 1) + (-1, 1, -1, 1) + (1, -1, -1, 1) \\
 & + (0, -1, 1) + (1, -2, 1) + (1, -1, 0) + (-1, 1, 0) + (0, 0).
 \end{aligned}$$

Positive-index map and stuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the stuffle algebra?

Positive-index map and stuffle product

Proposition (F.)

Let \mathbf{k}, \mathbf{k}' be two integer indices. Then,

$$m_{\mathbf{k}*\mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\}.$$

Proof sketch.

By induction on the sum of the depth of \mathbf{k} and \mathbf{k}' , we check the recursion condition in the definition of the stuffle product. □

Example

$$\begin{aligned} & m_{(1,-1)*(-1,1)} \\ &= \min\{-4, -4, -4, -3, -3, -4, -4, -4, -3, -3, -3, -3, -2\} = -4 \\ & \min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2, -2, -4\} = -4 \end{aligned}$$

Positive-index map and stuffle product

Corollary (F.)

The space $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}$ (resp. $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\}$) of admissible (resp. regularizable) integer indices is a \mathbb{Q} -subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$.

Proof.

Let \mathbf{k}, \mathbf{k}' be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\mathbf{k}*\mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$. □

Positive-index map and stuffle product

Theorem (F.)

Let

$\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, *)$, $\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, *)$
be the \mathbb{Q} -algebra with stuffle product $*$. Then, the positive-index map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}$$

is a \mathbb{Q} -algebra homomorphism, that is,

$$* \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ *.$$

Proof sketch.

By induction on the sum of the depth of \mathbf{k} and \mathbf{k}' , we check the recursion condition in the definition of the stuffle product. □

p-adic multiple zeta values

p -adic (single variable) multiple polylogarithms

Definition

Fix a prime p . For an admissible positive integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$, the p -adic (single variable) multiple polylogarithms are defined by the following series

$$\mathrm{Li}_{k_1, \dots, k_r}^p(z) = \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}} \in \mathbb{Q}_p[[z]].$$

Remark

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. Let π^+ be the positive-index map. Then,

$$\mathrm{Li}_{\mathbf{k}}^p(z) = \sum_{I \in \mathrm{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \mathrm{Li}_I^p(z)$$

holds for the coefficients $c_{\mathbf{k}, I} \in \mathbb{Q} \setminus \{0\}$.

p -adic multiple zeta values

Definition ([Furusho])

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$ be an admissible positive integer index. The p -adic multiple zeta values are defined by the following specific limit

$$\zeta_p(k_1, \dots, k_r) := \lim'_{z \rightarrow 1} \text{Li}_{k_1, \dots, k_r}^p(z) \in \mathbb{Q}_p.$$

Definition

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. We define the p -adic multiple zeta value $\zeta_p(k_1, \dots, k_r)$ for an integer index by

$$\begin{aligned} \zeta_p(k_1, \dots, k_r) &:= \lim'_{z \rightarrow 1} \text{Li}_{\mathbf{k}}^p(z) \\ &= \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \lim'_{z \rightarrow 1} \text{Li}_I^p(z) = \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \zeta_p(I) \in \mathbb{Q}_p, \end{aligned}$$

where π^+ is the positive-index map.

Double shuffle for p -adic MZVs of positive integer indices

We consider the \mathbb{Q} -linear map

$$\zeta_p : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \mathbb{Q}_p$$

defined by $\mathbf{k} \mapsto \zeta_p(\mathbf{k})$.

Remark ([Besser, Furusho], [Furusho, Jafari])

The p -adic multiple zeta values for admissible positive integer indices satisfy the double shuffle relation, that is, the \mathbb{Q} -linear map ζ_p is the \mathbb{Q} -algebra homomorphism with respect to the shuffle product \sqcup

$$\zeta_p : (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, \sqcup) \rightarrow \mathbb{Q}_p$$

and is the \mathbb{Q} -algebra homomorphism with respect to the stuffle product $*$

$$\zeta_p : (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, *) \rightarrow \mathbb{Q}_p.$$

Double shuffle for p -adic MZVs of integer indices

Proposition (F.)

Let $(\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, \sqcup)$ be the \mathbb{Q} -algebra of admissible integer indices. Then, the linear map

$$\zeta_p : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \mathbb{Q}_p$$

is a \mathbb{Q} -algebra homomorphism, i.e., $\zeta_p(\mathbf{k} \sqcup \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}')$.

Proposition (F.)

Let $(\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, *)$ be the \mathbb{Q} -algebra of admissible integer indices. Then, the linear map

$$\zeta_p : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \mathbb{Q}_p$$

is a \mathbb{Q} -algebra homomorphism, i.e., $\zeta_p(\mathbf{k} * \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}')$.

Double shuffle for p -adic MZVs of integer indices

Proof.

The composition of algebra homomorphism ζ_p and π^+ is still an algebra homomorphism. □

Theorem (F.)

Let \mathbf{k}, \mathbf{k}' be admissible integer indices. The p -adic multiple zeta values at integer indices satisfy the double shuffle relation, that is,

$$\zeta_p(\mathbf{k} \sqcup \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$

Proof.

By the two propositions above, one can see that

$$\zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} \sqcup \mathbf{k}') \text{ and } \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$

□

Example

Example

For admissible integer indices (a) and (b, c) with $b < 0$, we have

$$\begin{aligned} \zeta_p((a) \sqcup (b, c)) &= \sum_{i=0}^c \binom{a+i}{i} \zeta_p(b, c-i, a+i) + \\ &\sum_{i=0}^a \binom{c+i}{i} \left[\sum_{j=0}^{\min\{a-i-1, -b\}} (-1)^j \binom{-b}{j} \zeta_p(a-i-j, b+j, c+i) \right. \\ &\left. + (-1)^{a-i} \sum_{j=0}^{-b-a+i} \binom{-b-1-j}{a-i-1} \zeta_p(-j, b+a-i+j, c+i) \right], \end{aligned}$$

where we formally let $\binom{n}{-1} = \delta_{n,-1}$ and

$$\begin{aligned} &\zeta_p((a) * (b, c)) \\ &= \zeta_p(b, c, a) + \zeta_p(b, a, c) + \zeta_p(a, b, c) + \zeta_p(a+b, c) + \zeta_p(b, a+c). \end{aligned}$$

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Thank you for your attention!