Double shuffle relations for integer indices and their application to p-adic multiple zeta values

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MZVs

Multiple zeta values

Definition

For $r \in \mathbb{N}$, an r-tuple $(k_1, \ldots, k_r)$ in $\mathbb{N}^r$ is called an **positive integer index** and denoted by $\boldsymbol{k}$. For $r = 0$, the 0-tuple is an **positive integer index** defined as $\emptyset$. A positive integer indices $\boldsymbol{k}$ is called **admissible** if $k_r > 1$.

Definition

For admissible positive integer indices $\boldsymbol{k}$, the **multiple zeta values** are the real numbers defined by

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Remark

When $\boldsymbol{k} = (k)$, multiple zeta values $\zeta(k)$ are special values of the Riemann zeta function.

Iterated integral

Definition

Let $a_1, \ldots, a_k \in \{0,1\}$ with $a_1 = 1, a_k = 0$. The **iterated integral** is defined by

$$I(a_1, \ldots, a_k) = \int_{0<t_1<\cdots<t_k<1} \prod_{j=1}^{k} \frac{(-1)^{a_j} dt_j}{t_j - a_j}.$$

Remark

Multiple zeta values admit the iterated integral expression

$$\zeta(k_1, \ldots, k_r) = I(1, \underbrace{0, \cdots, 0}_{k_1 - 1 \text{ times}}, \ldots, 1, \underbrace{0, \cdots, 0}_{k_r - 1 \text{ times}}).$$

Example

$$\zeta(4) = I(1,0,0,0), \quad \zeta(2,3) = I(1,0,1,0,0).$$

Shuffle relation

Example

Using the iterated integral expression

$$\zeta(k_1,\dots,k_r) = I(1,\{0\}^{k_1-1},\dots,1,\{0\}^{k_r-1})$$

to calculate

$$\begin{aligned}
\zeta(2)\times\zeta(2) =& I(1,0)\times I(1,0)\\
=& \int_{0<x_1<x_2<1}\frac{dx_1}{1-x_1}\frac{dx_2}{x_2}\times\int_{0<y_1<y_2<1}\frac{dy_1}{1-y_1}\frac{dy_2}{y_2}\\
=& \int_{0<x_1<x_2<y_1<y_2<1}\frac{dx_1}{1-x_1}\cdots\frac{dy_2}{y_2}+\cdots+\int_{0<y_1<y_2<x_1<x_2<1}\frac{dy_1}{1-y_1}\cdots\frac{dx_2}{x_2}\\
=& I(1,0,1,0)+I(1,1,0,0)+I(1,1,0,0)+I(1,1,0,0)\\
&+I(1,1,0,0)+I(1,0,1,0)\\
=& 2\zeta(2,2)+4\zeta(1,3)
\end{aligned}$$

Stuffle relation

Example

Using the series expression

$$\zeta(k_1,\dots,k_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

to calculate

$$\begin{aligned}\zeta(2)\times\zeta(2)&=\sum_{0<n}\frac{1}{n^2}\times\sum_{0<m}\frac{1}{m^2}\\&=\sum_{0<n<m}\frac{1}{n^2m^2}+\sum_{0<n=m}\frac{1}{n^2m^2}+\sum_{0<m<n}\frac{1}{m^2n^2}\\&=\zeta(2,2)+\zeta(4)+\zeta(2,2).\end{aligned}$$

Double shuffle relation:

$$2\zeta(2,2)+\zeta(4)=\zeta(2)\times\zeta(2)=2\zeta(2,2)+4\zeta(1,3).$$

Integer indices

MZVs of integer indices

Remark

For positive integer indices $(k_1, \ldots, k_r)$ with $k_r = 1$, the series

$$\sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

will diverge. Although it diverges, one can define multiple zeta values through regularization and prove that it is compatible with double shuffle relations.

How about if the positive integer are integer?
Will it diverge or converge?
Can we define $\zeta(-1, 2)$, $\zeta(-1, 3)$ or $\zeta(-1, 4)$?

Integer indices

Definition

For $r \in \mathbb{N}$, an r-tuple $(k_1, \ldots, k_r)$ in $\mathbb{Z}^r$ is called an **integer index** and denoted by $\boldsymbol{k}$. For $r = 0$, the 0-tuple is an **integer index** defined as $\emptyset$.

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$,

- the **weight** of $\boldsymbol{k}$ is defined by $\mathrm{wt}(\boldsymbol{k}) := k_1 + \cdots + k_r$,
- the **depth** of $\boldsymbol{k}$ is defined by $\mathrm{dep}(\boldsymbol{k}) := r$,
- the **tail index** of $\boldsymbol{k}$ is defined by $\boldsymbol{k}_t := (k_t, \ldots, k_r) \in \mathbb{Z}^{r-t+1}$, where $t = 1, \ldots, r$.

Definition

For two integer indices $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ and $\boldsymbol{k}' = (k'_1, \ldots, k'_{r'}) \in \mathbb{Z}^{r'}$, we define the **concatenation** of $\boldsymbol{k}, \boldsymbol{k}'$ to be $(k_1, \ldots, k_r, k'_1, \ldots, k'_{r'}) \in \mathbb{Z}^{r+r'}$ and denote it by $(\boldsymbol{k}, \boldsymbol{k}')$.

Integer indices

Definition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define the **regularizability index** $m_{\boldsymbol{k}}$ of $\boldsymbol{k}$ by

$$\begin{aligned} m_{\boldsymbol{k}} &:= \min\left\{\mathrm{wt}(\boldsymbol{k}_t) - \mathrm{dep}(\boldsymbol{k}_t) \middle| t = 1, \ldots, r\right\} \\ &= \min\left\{\sum_{i=t}^{r}(k_i - 1) \middle| t = 1, \ldots, r\right\}. \end{aligned}$$

For $\boldsymbol{k} = \emptyset$, we define $m_{\emptyset} := \infty$.

Admissible and regularizable

Definition

An integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ is called **admissible** if $m_{\boldsymbol{k}} > 0$, is called **regularizable** if $m_{\boldsymbol{k}} \geq 0$.

Remark

For a positive integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, it is admissible if and only if $k_r > 1$, and it is always regularizable.

Proposition

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, it is admissible if and only if the series

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

converge, and it is regularizable if and only if one can define multiple zeta values through regularization.

Example

Example

Let $\boldsymbol{k} = (a, b)$ be an integer index. Then,

$$\mathrm{wt}(\boldsymbol{k}) = a + b, \ \mathrm{dep}(\boldsymbol{k}) = 2,$$

$$\boldsymbol{k}_1 = \boldsymbol{k} = (a, b), \ \boldsymbol{k}_2 = (b),$$

$$m_{\boldsymbol{k}} = \min\{a + b - 2, b - 1\}.$$

This implies $\boldsymbol{k}$ is admissible if $a + b - 2, b - 1 > 0$ and is regularizable if $a + b - 2, b - 1 \geq 0$.

Example

$\zeta(-1, 2)$ diverges.
$\zeta(-1, 3)$ diverges, but one can define it through regularization.
$\zeta(-1, 4)$ converges.

Admissible and regularizable

Definition

The set of (resp. positive) integer indices $\mathcal{I}$ (resp. $\mathcal{I}_{>0}$) is defined by

$$\mathcal{I} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r \text{ (resp. } \mathcal{I}_{>0} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r_{>0}).$$

Definition

The set of admissible (resp. positive) integer indices $\mathcal{I}^{\mathrm{adm}}$ (resp. $\mathcal{I}^{\mathrm{adm}}_{>0}$) is defined by

$$\mathcal{I}^{\mathrm{adm}} = \{\boldsymbol{k} \in \mathcal{I} | m_{\boldsymbol{k}} > 0\} \text{ (resp. } \mathcal{I}^{\mathrm{adm}}_{>0} = \{\boldsymbol{k} \in \mathcal{I}_{>0} | m_{\boldsymbol{k}} > 0\}).$$

The set of regularizable (resp. positive) integer indices $\mathcal{I}^{\mathrm{reg}}$ (resp. $\mathcal{I}^{\mathrm{reg}}_{>0}$) is defined by

$$\mathcal{I}^{\mathrm{reg}} = \{\boldsymbol{k} \in \mathcal{I} | m_{\boldsymbol{k}} \geq 0\} \text{ (resp. } \mathcal{I}^{\mathrm{reg}}_{>0} = \{\boldsymbol{k} \in \mathcal{I}_{>0} | m_{\boldsymbol{k}} \geq 0\}).$$

Admissible and regularizable

Definition

Let Σ be an element in the formal $\mathbb{Q}$-linear space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **support** $\mathrm{Supp}(\Sigma)$ of Σ to be the set of integer indices satisfying

$$\Sigma = \sum_{\boldsymbol{k} \in \mathrm{Supp}(\Sigma)} c_{\boldsymbol{k}} \boldsymbol{k}$$

with $c_{\boldsymbol{k}} \in \mathbb{Q} \setminus \{0\}$.

Definition

Let Σ be an element in the formal $\mathbb{Q}$-linear space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **regularizability index** m_Σ of Σ by

$$m_\Sigma := \min\left\{ m_{\boldsymbol{k}} \,\middle|\, \boldsymbol{k} \in \mathrm{Supp}(\Sigma) \right\}.$$

MPLs

Multiple polylogarithms

Definition

Let $\boldsymbol{k}$ be an integer index. The **(single variable) multiple polylogarithms** are the power series defined by

$$\mathrm{Li}_{\boldsymbol{k}}(z) := \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} \in \mathbb{Q}[\![z]\!].$$

Definition

We consider the following $\mathbb{Q}$-linear spaces of multiple polylogarithms:

1. $\mathcal{MPL}_{>0}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}_{>0}^{\mathrm{adm}}\right\} \subset \mathbb{Q}[\![z]\!]$.
2. $\mathcal{MPL}_{>0} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}_{>0}(=\mathcal{I}_{>0}^{\mathrm{reg}})\right\} \subset \mathbb{Q}[\![z]\!]$.
3. $\mathcal{MPL}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}^{\mathrm{adm}}\right\} \subset \mathbb{Q}[\![z]\!]$.
4. $\mathcal{MPL}^{\mathrm{reg}} = \mathrm{span}_{\mathbb{Q}}\left\{\mathrm{Li}_{\boldsymbol{k}}(z) \,\middle|\, \boldsymbol{k} \in \mathcal{I}^{\mathrm{reg}}\right\} \subset \mathbb{Q}[\![z]\!]$.

The proposition for MPLs

Proposition (F.)

The following equalities hold:

1. $\mathcal{MPL}^{\mathrm{adm}}_{>0} = \mathcal{MPL}^{\mathrm{adm}}$.
2. $\mathcal{MPL}_{>0}(= \mathcal{MPL}^{\mathrm{reg}}_{>0}) = \mathcal{MPL}^{\mathrm{reg}}$.

Corollary

For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, it is admissible if and only if the series

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

converge, and it is regularizable if and only if one can define multiple zeta values through regularization.

Faulhaber's formula

We denote by B_i be the Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note $B_1 = -\frac{1}{2}$.

Lemma (Faulhaber's formula)

Let k be a non-negative integer.

$$\sum_{n=1}^{m-1} n^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i m^{k+1-i} - \delta_{k,0}.$$

Proof of the proposition

Proposition (F.)

The following equalities hold:

1. $\mathcal{MPL}^{\mathrm{adm}}_{>0} = \mathcal{MPL}^{\mathrm{adm}}$.
2. $\mathcal{MPL}_{>0} (= \mathcal{MPL}^{\mathrm{reg}}_{>0}) = \mathcal{MPL}^{\mathrm{reg}}$.

Proof-Example.

Let $\boldsymbol{k} = (a, b)$ be an admissible integer index, i.e., $a + b - 2, b - 1 > 0$. We prove the multiple polylogarithm $\mathrm{Li}_{\boldsymbol{k}}(z) \in \mathcal{MPL}^{\mathrm{adm}}$ belongs to $\mathcal{MPL}^{\mathrm{adm}}_{>0}$. If $a > 0, b > 1$, then $\mathrm{Li}_{\boldsymbol{k}}(z) \in \mathcal{MPL}^{\mathrm{adm}}_{>0}$. If $a \leq 0, b > 1$, then by Faulhaber's formula we obtain

$$\mathrm{Li}_{\boldsymbol{k}}(z) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i \,\mathrm{Li}_{a+b-1+i}(z) - \delta_{-a,0} \,\mathrm{Li}_b(z) \in \mathcal{MPL}^{\mathrm{adm}}_{>0}.$$

One can inductively use Faulhaber's formula to prove this proposition. □

Positive-index map

Definition

The **positive-index map** is a $\mathbb{Q}$-linear map

$$\pi^+ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} \text{ (resp. } \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}_{>0}\})$$

defined by

$$\pi^+(\boldsymbol{k}) = \sum_{\boldsymbol{l}} c_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{l},$$

where the positive integer indices $\boldsymbol{l}$ and the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$ are uniquely obtained step by step through the inductive argument.

Example

Let $\boldsymbol{k} = (a, b)$ be an admissible integer index. If $a \leq 0, b > 1$, then

$$\pi^+(\boldsymbol{k}) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i(a+b-1+i) - \delta_{-a,0}(b).$$

Positive-index map

Remark

Actually, the positive-index map can be extended to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$, that is,

$$\pi^+ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, \qquad \pi^+(\boldsymbol{k}) = \sum_{\boldsymbol{l}} c_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{l},$$

where the integer indices

$$\boldsymbol{l} \in \{\emptyset\} \sqcup \bigsqcup_{r \in \mathbb{N}} \mathbb{Z}_{>0}^{r-1} \times \mathbb{Z}_{>m_{\boldsymbol{k}}}$$

and the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$ are obtained step by step through the same inductive argument. For integer indices $\boldsymbol{k} \in \mathcal{I}$, we have the identity

$$\mathrm{Li}_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} \, \mathrm{Li}_{\boldsymbol{l}}(z) \in \mathbb{Q}[\![z]\!].$$

Shuffle product

Shuffle product for integer indices

Definition

Let X be the alphabet $\{j, d, y\}$, and let W denote the set of words on the alphabet X, subject to the rule $jd = dj = \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. For an integer index $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define

$$w_{\boldsymbol{k}} = j^{k_r} y \cdots j^{k_1} y$$

to be the **corresponding word**. Conversely for $w = j^{k_r} y \cdots j^{k_1} y$, we define

$$\boldsymbol{k}_w = (k_1, \ldots, k_r)$$

to be the **corresponding integer index**.

Remark

We have a one-to-one correspondence between the set Wy and the set $\mathcal{I}$.

Shuffle product for integer indices

Definition ([Ebrahimi-Fard, Manchon, Singer])

We define the shuffle product by $\mathbf{1} \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, \mathbf{1} = w$ for any word $w \in \mathcal{W}y$, and recursively with respect to the sum of the lengths of two words in $\mathcal{W}y$:

1. $yu \sqcup\!\sqcup\, v := u \sqcup\!\sqcup\, yv := y(u \sqcup\!\sqcup\, v)$,
2. $ju \sqcup\!\sqcup\, jv := j(u \sqcup\!\sqcup\, jv) + j(ju \sqcup\!\sqcup\, v)$,
3. $du \sqcup\!\sqcup\, dv := d(u \sqcup\!\sqcup\, dv) - u \sqcup\!\sqcup\, d^2 v$,
4. $du \sqcup\!\sqcup\, jv := d(u \sqcup\!\sqcup\, jv) - u \sqcup\!\sqcup\, v$,
5. $ju \sqcup\!\sqcup\, dv := d(ju \sqcup\!\sqcup\, v) - u \sqcup\!\sqcup\, v$.

Definition

Let $\boldsymbol{k}, \boldsymbol{k}' \in \mathcal{I}$ be two integer indices. We define the shuffle product for integer indices by

$$\boldsymbol{k} \sqcup\!\sqcup\, \boldsymbol{k}' := \boldsymbol{k}_{w_{\boldsymbol{k}} \sqcup\!\sqcup\, w_{\boldsymbol{k}'}}.$$

Shuffle product for integer indices

Example

The corresponding words of $(1,-1),(-1,1)$ are $dyjy, jydy$, and

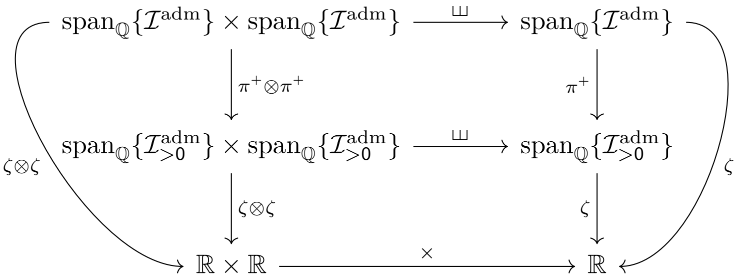
$$\begin{aligned}
& dyjy \sqcup\!\sqcup\, jydy \\
=& d(yjy \sqcup\!\sqcup\, jydy) - yjy \sqcup\!\sqcup\, ydy && \text{by } 4 \\
=& dy(jy \sqcup\!\sqcup\, jydy) - yy(jy \sqcup\!\sqcup\, dy) && \text{by } 1 \\
=& dy(j(y \sqcup\!\sqcup\, jydy) + j(jy \sqcup\!\sqcup\, ydy)) - yy(jy \sqcup\!\sqcup\, dy) && \text{by } 2 \\
=& dy(jyjydy + jy(jy \sqcup\!\sqcup\, dy)) - yy(jy \sqcup\!\sqcup\, dy) && \text{by } 1 \\
=& dyjyjydy + dyjy(d(jy \sqcup\!\sqcup\, y) - y \sqcup\!\sqcup\, y) - yy(d(jy \sqcup\!\sqcup\, y) - y \sqcup\!\sqcup\, y) && \text{by } 5 \\
=& dyjyjydy + dyjydyjy - dyjyyy - yydyjy + yyyy. && \text{by } 1
\end{aligned}$$

Hence, the shuffle product $(1,-1) \sqcup\!\sqcup\, (-1,1)$ of them is

$$(-1,1,1,-1) + (1,-1,1,-1) - (0,0,1,-1) - (1,-1,0,0) + (0,0,0,0).$$

Positive-index map and shuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the shuffle algebra?

Positive-index map and shuffle product

Proposition (F.)

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two integer indices. Then,

$$m_{\boldsymbol{k} ⧢ \boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\}.$$

Proof sketch.

By induction on the sum of the lengths of $w_{\boldsymbol{k}}$ and $w_{\boldsymbol{k}'}$, we check all recursion conditions in the definition of the shuffle product by considering cases according to the comparison between the index and 0. □

Example

$$\begin{aligned} m_{(1,-1) ⧢ (-1,1)} &= m_{(-1,1,1,-1)+(1,-1,1,-1)-(0,0,1,-1)-(1,-1,0,0)+(0,0,0,0)} \\ &= \min\{-4,-4,-4,-4,-4\} = -4 \end{aligned}$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2,-2,-4\} = -4$$

Positive-index map and shuffle product

Corollary (F.)

The space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}$ *(resp.* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\}$*) of admissible (resp. regularizable) integer indices is a* $\mathbb{Q}$*-subalgebra of* $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, ш)$*.*

Proof.

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\boldsymbol{k} ш \boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, ш)$. □

Positive-index map and shuffle product

Theorem (F.)

Let
$\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, ш), \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, ш)$
be the $\mathbb{Q}$-algebra with shuffle product ш. Then, the positive-index map

$$\pi^+ : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}$$

is a $\mathbb{Q}$-algebra homomorphism, that is,

$$ш \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ ш.$$

Proof sketch.

We show the general formula below by induction on the sum of the lengths of $w_{\mathbf{k}}$ and $w_{\mathbf{k}'}$.

$$\pi^+ \circ ш \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ ш : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}.$$ □

Stuffle product

Stuffle product for integer indices

We recall the stuffle product for integer indices.

Definition

We define the stuffle product

$$* : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}$$

recursively by:

$$\emptyset * \boldsymbol{k} = \boldsymbol{k} * \emptyset := \boldsymbol{k}$$

where $\boldsymbol{k} \in \mathcal{I}$ is an integer index, and for two integer indices $(\boldsymbol{k}, k), (\boldsymbol{k}', k')$ with $k, k' \in \mathbb{Z}$ and $\boldsymbol{k}, \boldsymbol{k}' \in \mathcal{I}$,

$$(\boldsymbol{k}, k) * (\boldsymbol{k}', k') := (\boldsymbol{k} * (\boldsymbol{k}', k'), k) + ((\boldsymbol{k}, k) * \boldsymbol{k}', k') + (\boldsymbol{k} * \boldsymbol{k}', k + k').$$

Stuffle product for integer indices

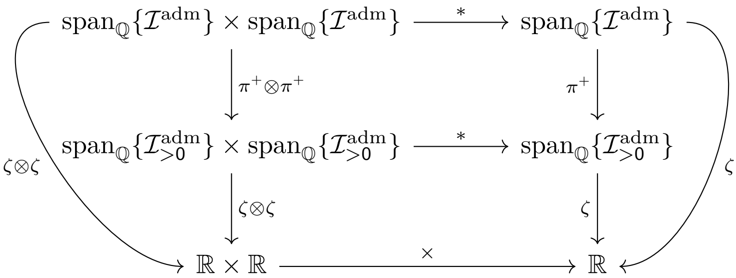
Example

The stuffle product of two integer indices $(1,-1)$ and $(-1,1)$ is

$$
\begin{aligned}
&(1,-1)*(-1,1)\\
=&((1)*(-1,1),-1)+((1,-1)*(-1),1)+((1)*(-1),-1+1)\\
=&((-1,1,1)+((1)*(-1),1)+(-1,2),-1)\\
&+(((1)*(-1),-1)+(1,-1,-1)+(1,-2),1)\\
&+((1,-1)+(-1,1)+(0),0)\\
=&((-1,1,1)+((1,-1)+(-1,1)+(0),1)+(-1,2),-1)\\
&+(((1,-1)+(-1,1)+(0),-1)+(1,-1,-1)+(1,-2),1)\\
&+(1,-1,0)+(-1,1,0)+(0,0)\\
=&(-1,1,1,-1)+(1,-1,1,-1)+(-1,1,1,-1)+(0,1,-1)\\
&+(-1,2,-1)+(1,-1,-1,1)+(-1,1,-1,1)+(1,-1,-1,1)\\
&+(0,-1,1)+(1,-2,1)+(1,-1,0)+(-1,1,0)+(0,0).
\end{aligned}
$$

Positive-index map and stuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the stuffle algebra?

Positive-index map and stuffle product

Proposition (F.)

Let $\mathbf{k}, \mathbf{k}'$ be two integer indices. Then,

$$m_{\mathbf{k}*\mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\}.$$

Proof sketch.

By induction on the sum of the depth of $\mathbf{k}$ and $\mathbf{k}'$, we check the recursion condition in the definition of the stuffle product. □

Example

$$\begin{aligned} &m_{(1,-1)*(-1,1)} \\ &= \min\{-4,-4,-4,-3,-3,-4,-4,-4,-3,-3,-3,-3,-2\} = -4 \end{aligned}$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2,-2,-4\} = -4$$

Positive-index map and stuffle product

Corollary (F.)

The space $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}$ *(resp.* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{reg}}\}$*) of admissible (resp. regularizable) integer indices is a* $\mathbb{Q}$*-subalgebra of* $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$*.*

Proof.

Let $\boldsymbol{k}, \boldsymbol{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\boldsymbol{k}*\boldsymbol{k}'} = \min\{m_{\boldsymbol{k}}, m_{\boldsymbol{k}'}, m_{\boldsymbol{k}} + m_{\boldsymbol{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$. □

Positive-index map and stuffle product

Theorem (F.)

Let
$\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, *), \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\} = (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, *)$
be the $\mathbb{Q}$-algebra with stuffle product $*$. Then, the positive-index map

$$\pi^{+} : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}$$

is a $\mathbb{Q}$-algebra homomorphism, that is,

$$* \circ (\pi^{+} \otimes \pi^{+}) = \pi^{+} \circ *.$$

Proof sketch.

By induction on the sum of the depth of $\boldsymbol{k}$ and $\boldsymbol{k}'$, we check the recursion condition in the definition of the stuffle product. □

Application

p-adic (single variable) multiple polylogarithms

Definition

Fix a prime p. For an admissible positive integer index $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$, the **$p$-adic (single variable) multiple polylogarithms** are defined by the following series

$$\mathrm{Li}^p_{k_1,\dots,k_r}(z) = \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} \in \mathbb{Q}_p[\![z]\!].$$

Remark

Let $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. Let π^+ be the positive-index map. Then,

$$\mathrm{Li}^p_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{l}\in\mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}}\, \mathrm{Li}^p_{\boldsymbol{l}}(z)$$

holds for the coefficients $c_{\boldsymbol{k},\boldsymbol{l}} \in \mathbb{Q} \setminus \{0\}$.

p-adic multiple zeta values

Definition ([Furusho])

Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ be an admissible positive integer index. The p**-adic multiple zeta values** are defined by the following specific limit

$$\zeta_p(k_1, \ldots, k_r) := {\lim_{z \to 1}}' \operatorname{Li}^p_{k_1, \ldots, k_r}(z) \in \mathbb{Q}_p.$$

Definition

Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. We define the p**-adic multiple zeta value** $\zeta_p(k_1, \ldots, k_r)$ for an integer index by

$$\zeta_p(k_1, \ldots, k_r) := {\lim_{z \to 1}}' \operatorname{Li}^p_{\boldsymbol{k}}(z)$$

$$= \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} {\lim_{z \to 1}}' \operatorname{Li}^p_{\boldsymbol{l}}(z) = \sum_{\boldsymbol{l} \in \mathrm{Supp}(\pi^+(\boldsymbol{k}))} c_{\boldsymbol{k},\boldsymbol{l}} \zeta_p(\boldsymbol{l}) \in \mathbb{Q}_p,$$

where π^+ is the positive-index map.

Double shuffle for p-adic MZVs of positive integer indices

We consider the $\mathbb{Q}$-linear map

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

defined by $\boldsymbol{k} \mapsto \zeta_p(\boldsymbol{k})$.

Remark ([Besser, Furusho], [Furusho, Jafari])

The p-adic multiple zeta values for admissible positive integer indices satisfy the double shuffle relation, that is, the $\mathbb{Q}$-linear map ζ_p is the $\mathbb{Q}$-algebra homomorphism with respect to the shuffle product $ш$

$$\zeta_p : (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, ш) \to \mathbb{Q}_p$$

and is the $\mathbb{Q}$-algebra homomorphism with respect to the stuffle product $*$

$$\zeta_p : (\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}_{>0}\}, *) \to \mathbb{Q}_p.$$

Double shuffle for p-adic MZVs of integer indices

Proposition (F.)

Let $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, ш)$ *be the* $\mathbb{Q}$*-algebra of admissible integer indices. Then, the linear map*

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

is a $\mathbb{Q}$*-algebra homomorphism, i.e.,* $\zeta_p(\boldsymbol{k} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \boldsymbol{k}') = \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}')$.

Proposition (F.)

Let $(\mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\}, *)$ *be the* $\mathbb{Q}$*-algebra of admissible integer indices. Then, the linear map*

$$\zeta_p : \mathrm{span}_{\mathbb{Q}}\{\mathcal{I}^{\mathrm{adm}}\} \to \mathbb{Q}_p$$

is a $\mathbb{Q}$*-algebra homomorphism, i.e.,* $\zeta_p(\boldsymbol{k} * \boldsymbol{k}') = \zeta_p(\boldsymbol{k})\zeta_p(\boldsymbol{k}')$.

Double shuffle for p-adic MZVs of integer indices

Proof.

The composition of algebra homomorphism ζ_p and π^+ is still an algebra homomorphism. □

Theorem (F.)

Let $\mathbf{k}, \mathbf{k}'$ be admissible integer indices. The p-adic multiple zeta values at integer indices satisfy the double shuffle relation, that is,

$$\zeta_p(\mathbf{k} ш \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$

Proof.

By the two propositions above, one can see that

$$\zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} ш \mathbf{k}') \text{ and } \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$

□

Example

Example

For admissible integer indices (a) and (b, c) with $b < 0$, we have

$$\zeta_p((a) ⧢ (b,c)) = \sum_{i=0}^{c} \binom{a+i}{i} \zeta_p(b, c-i, a+i) +$$

$$\sum_{i=0}^{a} \binom{c+i}{i} \left[\sum_{j=0}^{\min\{a-i-1,-b\}} (-1)^j \binom{-b}{j} \zeta_p(a-i-j, b+j, c+i) \right.$$

$$\left. +(-1)^{a-i} \sum_{j=0}^{-b-a+i} \binom{-b-1-j}{a-i-1} \zeta_p(-j, b+a-i+j, c+i) \right],$$

where we formally let $\binom{n}{-1} = \delta_{n,-1}$ and

$$\zeta_p((a) * (b,c))$$
$$= \zeta_p(b,c,a) + \zeta_p(b,a,c) + \zeta_p(a,b,c) + \zeta_p(a+b,c) + \zeta_p(b,a+c).$$

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Thank you for your attention!