

Double shuffle relations for p -adic multiple zeta values of integer indices

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MZVs

Multiple zeta values

Definition

For $r \in \mathbb{N}$, an r -tuple $(k_1, \dots, k_r)$ in $\mathbb{N}^r$ is called a **positive integer index** and denoted by $\mathbf{k}$. For $r = 0$, the 0-tuple is a **positive integer index** defined as $\emptyset$. A positive integer indices $\mathbf{k}$ is called **admissible** if $k_r > 1$.

Definition

For admissible positive integer indices $\mathbf{k}$, the **multiple zeta values** are the real numbers defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}.$$

Example

$$\zeta(1, 2) = \zeta(3).$$

MZVs of integer indices

Remark

For positive integer indices $(k_1, \dots, k_r)$ with $k_r = 1$, the series

$$\zeta(k_1, \dots, k_r) = \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

will diverge. Although it diverges, one can define multiple zeta values through regularization and prove that the regularized multiple zeta values is compatible with double shuffle relations.

What happens if some k_i are zero or negative?

Does this series converge or diverge?

Can we define $\zeta(-1, 2)$, $\zeta(-1, 3)$ or $\zeta(-1, 4)$?

p -adic (single variable) multiple polylogarithms

Definition

Fix a prime p . For an admissible positive integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, the **p -adic (single variable) multiple polylogarithms** are defined by the following series

$$\mathrm{Li}_{k_1, \dots, k_r}^p(z) = \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}} \in \mathbb{Q}_p[[z]].$$

Definition ([Furusho])

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$ be an admissible positive integer index. The **p -adic multiple zeta values** are defined by the following specific limit

$$\zeta_p(k_1, \dots, k_r) := \lim_{z \rightarrow 1}' \mathrm{Li}_{k_1, \dots, k_r}^p(z) \in \mathbb{Q}_p.$$

Integer indices

Definition

For $r \in \mathbb{N}$, an r -tuple $(k_1, \dots, k_r)$ in $\mathbb{Z}^r$ is called an **integer index** and denoted by $\mathbf{k}$. For $r = 0$, the 0-tuple is an **integer index** defined as $\emptyset$.

Definition

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$,

- the **weight** of $\mathbf{k}$ is defined by $\text{wt}(\mathbf{k}) := k_1 + \dots + k_r$,
- the **depth** of $\mathbf{k}$ is defined by $\text{dep}(\mathbf{k}) := r$,
- the **tail index** of $\mathbf{k}$ is defined by $\mathbf{k}_t := (k_t, \dots, k_r) \in \mathbb{Z}^{r-t+1}$, where $t = 1, \dots, r$.

Definition

For two integer indices $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $\mathbf{k}' = (k'_1, \dots, k'_{r'}) \in \mathbb{Z}^{r'}$, we define the **concatenation** of $\mathbf{k}, \mathbf{k}'$ to be $(k_1, \dots, k_r, k'_1, \dots, k'_{r'}) \in \mathbb{Z}^{r+r'}$ and denote it by $(\mathbf{k}, \mathbf{k}')$.

Integer indices

Definition

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we define the **regularizability index** $m_{\mathbf{k}}$ of $\mathbf{k}$ by

$$\begin{aligned} m_{\mathbf{k}} &:= \min \{ \text{wt}(\mathbf{k}_t) - \text{dep}(\mathbf{k}_t) \mid t = 1, \dots, r \} \\ &= \min \left\{ \sum_{i=t}^r (k_i - 1) \mid t = 1, \dots, r \right\}. \end{aligned}$$

For $\mathbf{k} = \emptyset$, we define $m_{\emptyset} := \infty$.

Admissible and regularizable

Definition

An integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ is called **admissible** if $m_{\mathbf{k}} > 0$, is called **regularizable** if $m_{\mathbf{k}} \geq 0$.

Remark

For a positive integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, it is admissible if and only if $k_r > 1$, and it is always regularizable.

Proposition

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, it is admissible if and only if the series

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

converge, and it is regularizable if and only if one can define multiple zeta values through regularization.

Admissible and regularizable

Example

Let $\mathbf{k} = (a, b)$ be an integer index. Then,

$$\text{wt}(\mathbf{k}) = a + b, \quad \text{dep}(\mathbf{k}) = 2,$$

$$\mathbf{k}_1 = \mathbf{k} = (a, b), \quad \mathbf{k}_2 = (b),$$

$$m_{\mathbf{k}} = \min\{a + b - 2, b - 1\}.$$

This implies $\mathbf{k}$ is admissible if $a + b - 2, b - 1 > 0$ and is regularizable if $a + b - 2, b - 1 \geq 0$.

Example

$\zeta(-1, 2)$ diverges.

$\zeta(-1, 3)$ diverges, but one can define it through regularization.

$\zeta(-1, 4)$ converges.

Admissible and regularizable

Definition

The set of (resp. positive) integer indices $\mathcal{I}$ (resp. $\mathcal{I}_{>0}$) is defined by

$$\mathcal{I} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}^r \quad (\text{resp. } \mathcal{I}_{>0} = \bigsqcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{>0}^r).$$

The set of admissible (resp. positive) integer indices $\mathcal{I}^{\text{adm}}$ (resp. $\mathcal{I}_{>0}^{\text{adm}}$) is defined by

$$\mathcal{I}^{\text{adm}} = \{\mathbf{k} \in \mathcal{I} \mid m_{\mathbf{k}} > 0\} \quad (\text{resp. } \mathcal{I}_{>0}^{\text{adm}} = \{\mathbf{k} \in \mathcal{I}_{>0} \mid m_{\mathbf{k}} > 0\}).$$

The set of regularizable (resp. positive) integer indices $\mathcal{I}^{\text{reg}}$ (resp. $\mathcal{I}_{>0}^{\text{reg}}$) is defined by

$$\mathcal{I}^{\text{reg}} = \{\mathbf{k} \in \mathcal{I} \mid m_{\mathbf{k}} \geq 0\} \quad (\text{resp. } \mathcal{I}_{>0}^{\text{reg}} = \{\mathbf{k} \in \mathcal{I}_{>0} \mid m_{\mathbf{k}} \geq 0\}).$$

Admissible and regularizable

Definition

Let Σ be an element in the formal $\mathbb{Q}$ -linear space $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **support** $\text{Supp}(\Sigma)$ of Σ to be the set of integer indices satisfying

$$\Sigma = \sum_{\mathbf{k} \in \text{Supp}(\Sigma)} c_{\mathbf{k}} \mathbf{k}$$

with $c_{\mathbf{k}} \in \mathbb{Q} \setminus \{0\}$.

Definition

Let Σ be an element in the formal $\mathbb{Q}$ -linear space $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$. We define the **regularizability index** m_{Σ} of Σ by

$$m_{\Sigma} := \min \{m_{\mathbf{k}} \mid \mathbf{k} \in \text{Supp}(\Sigma)\}.$$

MPLs

Multiple polylogarithms

Definition

Let $\mathbf{k}$ be an integer index. The **(single variable) multiple polylogarithms** are the power series defined by

$$\mathrm{Li}_{\mathbf{k}}(z) := \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}} \in \mathbb{Q}[[z]].$$

Definition

We consider the following $\mathbb{Q}$ -linear spaces of multiple polylogarithms:

- ① $\mathcal{MPL}_{>0}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}_{>0}^{\mathrm{adm}} \} \subset \mathbb{Q}[[z]].$
- ② $\mathcal{MPL}_{>0} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}_{>0} (= \mathcal{I}_{>0}^{\mathrm{reg}}) \} \subset \mathbb{Q}[[z]].$
- ③ $\mathcal{MPL}^{\mathrm{adm}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}^{\mathrm{adm}} \} \subset \mathbb{Q}[[z]].$
- ④ $\mathcal{MPL}^{\mathrm{reg}} = \mathrm{span}_{\mathbb{Q}} \{ \mathrm{Li}_{\mathbf{k}}(z) \mid \mathbf{k} \in \mathcal{I}^{\mathrm{reg}} \} \subset \mathbb{Q}[[z]].$

The proposition for MPLs

Proposition (F.)

The following equalities hold:

- ① $\mathcal{MPL}_{>0}^{\text{adm}} = \mathcal{MPL}^{\text{adm}}.$
- ② $\mathcal{MPL}_{>0}(= \mathcal{MPL}_{>0}^{\text{reg}}) = \mathcal{MPL}^{\text{reg}}.$

Corollary

For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, it is admissible if and only if the series

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

converge, and it is regularizable if and only if one can define multiple zeta values through regularization.

Faulhaber's formula

We denote by B_i be the Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

We note $B_1 = -\frac{1}{2}$.

Lemma (Faulhaber's formula)

Let k be a non-negative integer.

$$\sum_{n=1}^{m-1} n^k = \frac{1}{k+1} \sum_{i=0}^k \binom{k+1}{i} B_i m^{k+1-i} - \delta_{k,0}.$$

Proof of the proposition

Proposition (F.)

The following equalities hold:

- ① $\mathcal{MPL}_{>0}^{\text{adm}} = \mathcal{MPL}^{\text{adm}}.$
- ② $\mathcal{MPL}_{>0}(= \mathcal{MPL}_{>0}^{\text{reg}}) = \mathcal{MPL}^{\text{reg}}.$

Proof-Example.

Let $\mathbf{k} = (a, b)$ be an admissible integer index, i.e., $a + b - 2, b - 1 > 0$. We prove the multiple polylogarithm $\text{Li}_{\mathbf{k}}(z) \in \mathcal{MPL}^{\text{adm}}$ belongs to $\mathcal{MPL}_{>0}^{\text{adm}}$. If $a > 0, b > 1$, then $\text{Li}_{\mathbf{k}}(z) \in \mathcal{MPL}_{>0}^{\text{adm}}$. If $a \leq 0, b > 1$, then by Faulhaber's formula we obtain

$$\text{Li}_{\mathbf{k}}(z) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i \text{Li}_{a+b-1+i}(z) - \delta_{-a,0} \text{Li}_b(z) \in \mathcal{MPL}_{>0}^{\text{adm}}.$$

One can inductively use Faulhaber's formula to prove this proposition. □

Positive-index map

Definition

The **positive-index map** is a $\mathbb{Q}$ -linear map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} \text{ (resp. } \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{reg}}\})$$

defined by

$$\pi^+(\mathbf{k}) = \sum_I c_{\mathbf{k},I} I,$$

where the positive integer indices I and the coefficients $c_{\mathbf{k},I} \in \mathbb{Q} \setminus \{0\}$ are uniquely obtained step by step through the inductive argument.

Example

Let $\mathbf{k} = (a, b)$ be an admissible integer index. If $a \leq 0, b > 1$, then

$$\pi^+(\mathbf{k}) = \frac{1}{-a+1} \sum_{i=0}^{-a} \binom{-a+1}{i} B_i(a+b-1+i) - \delta_{-a,0}(b).$$

Positive-index map

Remark

Actually, the positive-index map can be extended to $\text{span}_{\mathbb{Q}}\{\mathcal{I}\}$, that is,

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \quad \pi^+(\mathbf{k}) = \sum_I c_{\mathbf{k},I} I,$$

where the integer indices

$$I \in \{\emptyset\} \sqcup \bigsqcup_{r \in \mathbb{N}} \mathbb{Z}_{>0}^{r-1} \times \mathbb{Z}_{>m_k}$$

and the coefficients $c_{\mathbf{k},I} \in \mathbb{Q} \setminus \{0\}$ are obtained step by step through the same inductive argument. For integer indices $\mathbf{k} \in \mathcal{I}$, we have the identity

$$\text{Li}_{\mathbf{k}}(z) = \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k},I} \text{Li}_I(z) \in \mathbb{Q}[[z]].$$

Shuffle product

Shuffle product for integer indices

Definition

Let X be the alphabet $\{j, d, y\}$, and let W denote the set of words on the alphabet X , subject to the rule $jd = dj = \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. For an integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we define

$$w_{\mathbf{k}} = j^{k_r} y \cdots j^{k_1} y$$

to be the **corresponding word**. Conversely for $w = j^{k_r} y \cdots j^{k_1} y$, we define

$$\mathbf{k}_w = (k_1, \dots, k_r)$$

to be the **corresponding integer index**.

Remark

We have a one-to-one correspondence between the set W and the set $\mathcal{I}$.

Shuffle product for integer indices

Definition ([Ebrahimi-Fard, Manchon, Singer])

We define the shuffle product by $\mathbf{1} \sqcup w = w \sqcup \mathbf{1} = w$ for any word $w \in Wy$, and recursively with respect to the sum of the lengths of two words in Wy :

- ① $yu \sqcup v := u \sqcup yv := y(u \sqcup v),$
- ② $ju \sqcup jv := j(u \sqcup jv) + j(ju \sqcup v),$
- ③ $du \sqcup dv := d(u \sqcup dv) - u \sqcup d^2v,$
- ④ $du \sqcup jv := d(u \sqcup jv) - u \sqcup v,$
- ⑤ $ju \sqcup dv := d(ju \sqcup v) - u \sqcup v.$

Definition

Let $k, k' \in \mathcal{I}$ be two integer indices. We define the shuffle product for integer indices by

$$k \sqcup k' := k_{w_k \sqcup w_{k'}}.$$

Shuffle product for integer indices

Example

The corresponding words of $(1, -1)$, $(-1, 1)$ are $dyjy$, $jydy$, and

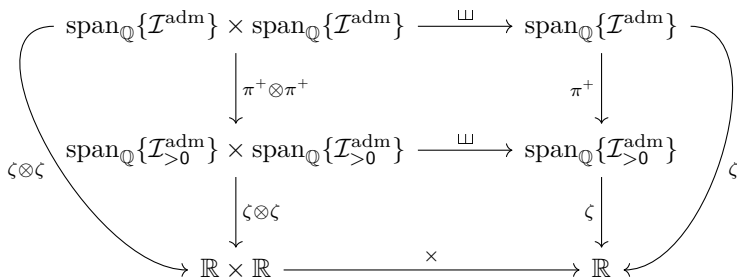
$$\begin{aligned}
 & dyjy \sqcup jydy \\
 = & d(yjy \sqcup jydy) - yjy \sqcup ydy && \text{by 4} \\
 = & dy(jy \sqcup jydy) - yy(jy \sqcup dy) && \text{by 1} \\
 = & dy(j(y \sqcup jydy) + j(jy \sqcup ydy)) - yy(jy \sqcup dy) && \text{by 2} \\
 = & dy(jyjydy + jy(jy \sqcup dy)) - yy(jy \sqcup dy) && \text{by 1} \\
 = & dyjyjydy + dyjy(d(jy \sqcup y) - y \sqcup y) - yy(d(jy \sqcup y) - y \sqcup y) && \text{by 5} \\
 = & dyjyjydy + dyjydyjy - dyjyyy - yydyjy + yyyy. && \text{by 1}
 \end{aligned}$$

Hence, the shuffle product $(1, -1) \sqcup (-1, 1)$ of them is

$$(-1, 1, 1, -1) + (1, -1, 1, -1) - (0, 0, 1, -1) - (1, -1, 0, 0) + (0, 0, 0, 0).$$

Positive-index map and shuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the shuffle algebra?

Positive-index map and shuffle product

Proposition (F.)

Let k, k' be two integer indices. Then,

$$m_{k \sqcup k'} = \min\{m_k, m_{k'}, m_k + m_{k'}\}.$$

Proof sketch.

By induction on the sum of the lengths of two words, we check all recursion conditions in the definition of the shuffle product by considering cases according to comparison between regularizability index and 0. $\square$

Example

$$\begin{aligned} m_{(1,-1) \sqcup (-1,1)} &= m_{(-1,1,1,-1)+(1,-1,1,-1)-(0,0,1,-1)-(1,-1,0,0)+(0,0,0,0)} \\ &= \min\{-4, -4, -4, -4, -4\} = -4 \end{aligned}$$

$$\min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2, -2, -4\} = -4$$

Positive-index map and shuffle product

Corollary (F.)

The space $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}$ (resp. $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\}$) of admissible (resp. regularizable) integer indices is a $\mathbb{Q}$ -subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \sqcup)$.

Proof.

Let $\mathbf{k}, \mathbf{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\mathbf{k} \sqcup \mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\} > 0 \text{ (resp. } \geq 0).$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, \sqcup)$. □

Positive-index map and shuffle product

Theorem (F.)

Let

$\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, \sqcup), \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, \sqcup)$
be the $\mathbb{Q}$ -algebra with shuffle product $\sqcup$. Then, the positive-index map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}$$

is a $\mathbb{Q}$ -algebra homomorphism, that is,

$$\sqcup \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ \sqcup.$$

Proof sketch.

We show the general formula below by induction on the sum of the lengths of w_k and $w_{k'}$.

$$\pi^+ \circ \sqcup \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ \sqcup : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}. \quad \square$$

Shuffle product for integer indices

We recall the shuffle product for integer indices.

Definition

We define the shuffle product

$$* : \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \times \text{span}_{\mathbb{Q}}\{\mathcal{I}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}\}$$

recursively by:

$$\emptyset * \mathbf{k} = \mathbf{k} * \emptyset := \mathbf{k}$$

where $\mathbf{k} \in \mathcal{I}$ is an integer index, and for two integer indices $(\mathbf{k}, k), (\mathbf{k}', k')$ with $k, k' \in \mathbb{Z}$ and $\mathbf{k}, \mathbf{k}' \in \mathcal{I}$,

$$(\mathbf{k}, k) * (\mathbf{k}', k') := (\mathbf{k} * (\mathbf{k}', k'), k) + ((\mathbf{k}, k) * \mathbf{k}', k') + (\mathbf{k} * \mathbf{k}', k + k').$$

Shuffle product for integer indices

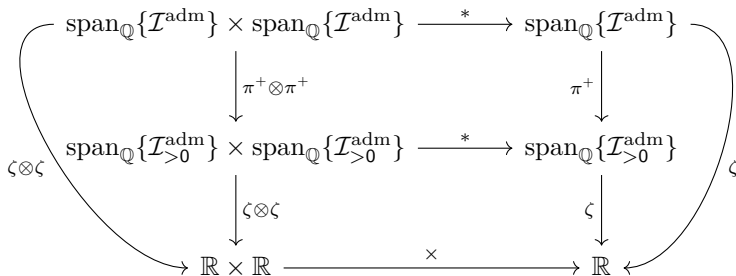
Example

The shuffle product of two integer indices $(1, -1)$ and $(-1, 1)$ is

$$\begin{aligned}
 & (1, -1) * (-1, 1) \\
 &= ((1) * (-1, 1), -1) + ((1, -1) * (-1), 1) + ((1) * (-1), -1 + 1) \\
 &= ((-1, 1, 1) + ((1) * (-1), 1) + (-1, 2), -1) \\
 &\quad + (((1) * (-1), -1) + (1, -1, -1) + (1, -2), 1) \\
 &\quad + ((1, -1) + (-1, 1) + (0), 0) \\
 &= ((-1, 1, 1) + ((1, -1) + (-1, 1) + (0), 1) + (-1, 2), -1) \\
 &\quad + (((1, -1) + (-1, 1) + (0), -1) + (1, -1, -1) + (1, -2), 1) \\
 &\quad + (1, -1, 0) + (-1, 1, 0) + (0, 0) \\
 &= (-1, 1, 1, -1) + (1, -1, 1, -1) + (-1, 1, 1, -1) + (0, 1, -1) \\
 &\quad + (-1, 2, -1) + (1, -1, -1, 1) + (-1, 1, -1, 1) + (1, -1, -1, 1) \\
 &\quad + (0, -1, 1) + (1, -2, 1) + (1, -1, 0) + (-1, 1, 0) + (0, 0).
 \end{aligned}$$

Positive-index map and stuffle product

Is this diagram commutative?



Equivalently, is the positive-index map π^+ is an algebra homomorphism between the stuffle algebra?

Positive-index map and stuffle product

Proposition (F.)

Let $\mathbf{k}, \mathbf{k}'$ be two integer indices. Then,

$$m_{\mathbf{k} * \mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\}.$$

Proof sketch.

By induction on the sum of the depth of $\mathbf{k}$ and $\mathbf{k}'$, we check the recursion condition in the definition of the stuffle product. □

Example

$$\begin{aligned} & m_{(1,-1)*(-1,1)} \\ &= \min\{-4, -4, -4, -3, -3, -4, -4, -4, -3, -3, -3, -3, -2\} = -4 \\ & \min\{m_{(1,-1)}, m_{(-1,1)}, m_{(1,-1)} + m_{(-1,1)}\} = \min\{-2, -2, -4\} = -4 \end{aligned}$$

Positive-index map and stuffle product

Corollary (F.)

*The space $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}$ (resp. $\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{reg}}\}$) of admissible (resp. regularizable) integer indices is a $\mathbb{Q}$ -subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$.*

Proof.

Let $\mathbf{k}, \mathbf{k}'$ be two admissible (resp. regularizable) integer indices. By the proposition, we have

$$m_{\mathbf{k}*\mathbf{k}'} = \min\{m_{\mathbf{k}}, m_{\mathbf{k}'}, m_{\mathbf{k}} + m_{\mathbf{k}'}\} > 0 \text{ (resp. } \geq 0\text{)}.$$

This shows that it is closed under the shuffle product, hence we have a subalgebra of $(\text{span}_{\mathbb{Q}}\{\mathcal{I}\}, *)$. □

Positive-index map and stuffle product

Theorem (F.)

Let

$\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, *)$, $\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\} = (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, *)$
be the $\mathbb{Q}$ -algebra with stuffle product $*$. Then, the positive-index map

$$\pi^+ : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}$$

is a $\mathbb{Q}$ -algebra homomorphism, that is,

$$* \circ (\pi^+ \otimes \pi^+) = \pi^+ \circ *.$$

Proof sketch.

By induction on the sum of the depth of $\mathbf{k}$ and $\mathbf{k}'$, we check the recursion condition in the definition of the stuffle product. □

p -adic MZVs

p -adic (single variable) multiple polylogarithms

Definition

Fix a prime p . For an admissible positive integer index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$, the p -adic (single variable) multiple polylogarithms are defined by the following series

$$\mathrm{Li}_{k_1, \dots, k_r}^p(z) = \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}} \in \mathbb{Q}_p[[z]].$$

Remark

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. Let π^+ be the positive-index map. Then,

$$\mathrm{Li}_{\mathbf{k}}^p(z) = \sum_{I \in \mathrm{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \mathrm{Li}_I^p(z)$$

holds for the coefficients $c_{\mathbf{k}, I} \in \mathbb{Q} \setminus \{0\}$.

p -adic multiple zeta values

Definition ([Furusho])

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$ be an admissible positive integer index. The **p -adic multiple zeta values** are defined by the following specific limit

$$\zeta_p(k_1, \dots, k_r) := \lim'_{z \rightarrow 1} \text{Li}_{k_1, \dots, k_r}^p(z) \in \mathbb{Q}_p.$$

Definition

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ be an admissible integer index. We define the **p -adic multiple zeta values** $\zeta_p(k_1, \dots, k_r)$ for integer indices by

$$\begin{aligned} \zeta_p(k_1, \dots, k_r) &:= \lim'_{z \rightarrow 1} \text{Li}_{\mathbf{k}}^p(z) \\ &= \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \lim'_{z \rightarrow 1} \text{Li}_I^p(z) = \sum_{I \in \text{Supp}(\pi^+(\mathbf{k}))} c_{\mathbf{k}, I} \zeta_p(I) \in \mathbb{Q}_p, \end{aligned}$$

where π^+ is the positive-index map.

Double shuffle for p -adic MZVs of positive integer indices

We consider the $\mathbb{Q}$ -linear map

$$\zeta_p : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \mathbb{Q}_p$$

defined by $\mathbf{k} \mapsto \zeta_p(\mathbf{k})$.

Remark ([Besser, Furusho], [Furusho, Jafari])

The p -adic multiple zeta values for admissible positive integer indices satisfy the double shuffle relation, that is, the $\mathbb{Q}$ -linear map ζ_p is the $\mathbb{Q}$ -algebra homomorphism with respect to the shuffle product $\sqcup$

$$\zeta_p^+ : (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, \sqcup) \rightarrow \mathbb{Q}_p$$

and is the $\mathbb{Q}$ -algebra homomorphism with respect to the stuffle product $*$

$$\zeta_p^+ : (\text{span}_{\mathbb{Q}}\{\mathcal{I}_{>0}^{\text{adm}}\}, *) \rightarrow \mathbb{Q}_p,$$

where $+$ means the restriction on positive integer indices.

Double shuffle for p -adic MZVs of integer indices

Proposition (F.)

Let $(\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, \sqcup)$ and $(\text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\}, *)$ be the $\mathbb{Q}$ -algebra of admissible integer indices. Then, the linear map

$$\zeta_p : \text{span}_{\mathbb{Q}}\{\mathcal{I}^{\text{adm}}\} \rightarrow \mathbb{Q}_p$$

is a $\mathbb{Q}$ -algebra homomorphism with respect to two products, i.e.,

$$\zeta_p(\mathbf{k} \sqcup \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}')$$

and

$$\zeta_p(\mathbf{k} * \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}').$$

Proof.

By definition, we have $\zeta_p = \zeta_p^+ \circ \pi^+$, and the composition of algebra homomorphism is an algebra homomorphism. □

Double shuffle for p -adic MZVs of integer indices

Theorem (F.)

Let $\mathbf{k}, \mathbf{k}'$ be admissible integer indices. The p -adic multiple zeta values at integer indices satisfy the double shuffle relation, that is,

$$\zeta_p(\mathbf{k} \sqcup \mathbf{k}') = \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$

Proof.

By the two propositions above, one can see that

$$\zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} \sqcup \mathbf{k}') \text{ and } \zeta_p(\mathbf{k})\zeta_p(\mathbf{k}') = \zeta_p(\mathbf{k} * \mathbf{k}').$$



Example

Example

For admissible integer indices (a) and (b, c) with $b < 0$, we have

$$\begin{aligned} \zeta_p((a) \sqcup (b, c)) &= \sum_{i=0}^c \binom{a+i}{i} \zeta_p(b, c-i, a+i) + \\ &\sum_{i=0}^a \binom{c+i}{i} \left[\sum_{j=0}^{\min\{a-i-1, -b\}} (-1)^j \binom{-b}{j} \zeta_p(a-i-j, b+j, c+i) \right. \\ &\left. + (-1)^{a-i} \sum_{j=0}^{-b-a+i} \binom{-b-1-j}{a-i-1} \zeta_p(-j, b+a-i+j, c+i) \right], \end{aligned}$$

where we formally let $\binom{n}{-1} = \delta_{n,-1}$ and

$$\begin{aligned} &\zeta_p((a) * (b, c)) \\ &= \zeta_p(b, c, a) + \zeta_p(b, a, c) + \zeta_p(a, b, c) + \zeta_p(a+b, c) + \zeta_p(b, a+c). \end{aligned}$$

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Thank you for your attention!